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Tables of Integrals of Complex-valued Functions of p-Adic Arguments

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Part I
Some Facts from p-Adic Analysis

Everywhere henceforth we shall assume, unless otherwise stipulated, that p takes all prime numbers, $p = 2, 3, 5, \dots, 137, \dots$, and γ takes all integer (rational) numbers, $\gamma = 0, \pm 1, \pm 2, \dots$, $\gamma \in Z$; By Z_+ we shall denote the set of natural numbers $\gamma = 1, 2, \dots$. If $\mathbb{K}$ is some field (or ring), by $\mathbb{K}^\times$ we shall denote its multiplicative group.

§1. The Field of p-Adic Numbers $\mathbb{Q}_p$

Denote: by Q the field of rational numbers, by $\mathbb{R}$ the field of real numbers, by $\mathbb{C}$ the field of complex numbers.

Let p be a prime number. Any rational number $x \neq 0$ uniquely represented in the form

$$x = \pm p^\gamma a/b$$

where $\gamma \in Z$ and a, b are natural numbers not divisible by p and without common divisors. *p-Adic norm* $|x|_p$ of the number $x \in \mathbb{Q}$ is defined by the formulas

$$|x|_p = p^{-\gamma}, x \neq 0, \quad |0|_p = 0.$$

The completion of the field $\mathbb{Q}$ with respect to the norm $|\cdot|_p$ is the *field of p-adic numbers* $\mathbb{Q}_p$.

The canonical form of a p-adic number $x \neq 0$ is

$$x = p^\gamma (x_0 + x_1 p + x_2 p^2 + \dots) \tag{1.1}$$

where $\gamma = \gamma(x) \in Z$, $x_j = 0, 1, \dots, p-1, x_0 \neq 0, j = 0, 1, \dots,$ besides $|x|_p = p^{-\gamma}$. The number $-\gamma$ is called *the order* of number x and it is denoted by $\operatorname{ord} x$, $\quad \operatorname{ord} x = -\gamma(x)$, $\quad \operatorname{ord} 0 = -\infty$.

The norm $|\cdot|_p$ possesses the following characteristic properties:

$$1) |x|_p \geqslant 0, \quad |x|_p = 0 \leftrightarrow x = 0,$$

$$2) |xy|_p = |x|_p |y|_p,$$

$$3) |x + y|_p \leqslant \max(|x|_p, |y|_p). \tag{1.2}$$

Besides,

$$3') |x + y|_p = \max(|x|_p, |y|_p), \quad |x|_p \neq |y|_p,$$

$$3'')|x+y|_p \leqslant |2x|_p, \quad |x|_p = |y|_p.$$

Thus, owing to (1.2), the norm $|\cdot|_p$ is *non-Archimedean* and the space $\mathbb{Q}_p$ is *ultrametric.*

Denote: by

$$B_\gamma(a) = [x \in \mathbb{Q}_p : |x-a|_p \leqslant p^\gamma]$$

a *disk* with a center at the point $a \in \mathbb{Q}_p$ of radius p^γ, $B_\gamma = B_\gamma(0)$; by

$$S_\gamma(a) = [x \in \mathbb{Q}_p : |x-a|_p = p^\gamma]$$

a *circumference* with the same center and radius, $S_\gamma = S_\gamma(0)$.

Obvious relations are valid:

$$B_\gamma(a) = \cup_{\gamma' \leqslant \gamma} S_{\gamma'}(a), \quad S_\gamma(a) = B_\gamma(a) \backslash B_{\gamma-1}(a),$$

$$\mathbb{Q}_p = \cup_{\gamma \in Z} B_\gamma(a), \quad \mathbb{Q}_p^\times = \cup_{\gamma \in Z} S_\gamma(a).$$

The geometry of the space $\mathbb{Q}_p$ is very unusual: all triangles in it are isosceles; every point of a disk is its center; a disk has no boundary; a disk is a finite union of disjoint disks of smaller radii; if two disks have a common point, so one of them is contained in another; a disk is open compact.

A set of $\mathbb{Q}_p$ which is closed and open is called *clopen* set.

Denote: by $Z_p = B_0$ the maximal compact subring of the field $\mathbb{Q}_p$ (the ring of integer p-adic numbers); by $Z_p^\times = S_0$ multiplicative group of the ring Z_p (it is the group of unities of the field $\mathbb{Q}_p$); by $I_p = pZ_p = B_{-1}$ maximal ideal of the ring Z_p.

The residue classes Z_p/I_p form the finite field which is isomorphic to the residue classes module $p : \{0, 1, \ldots, p-1\}$.

Introduce special sets:

$$G_p = [x \in \mathbb{Q}_p : |x|_p \leqslant |2p|_p];$$

$$J_p = [x \in Z_p^\times : 1 - x \in G_p],$$

J_p is a multiplicative group;

$$S_{\gamma, k_0 k_1 \ldots k_n} = [x \in S_\gamma : x_0 = k_0, x_1 = k_1, \ldots, x_n = k_n];$$

$$S_\gamma^{k_0 k_1 \ldots k_n} = [x \in S_\gamma : x_0 \neq k_0, x_1 \neq k_1, \ldots, x_n \neq k_n],$$

where $k_j = 0, 1, \ldots, p-1, k_0 \neq 0, j = 1, 2, \ldots, n$.

The sets just introduced are open compacts in $\mathbb{Q}_p$.

Rational part $\{x\}_p$ of a number $x \in \mathbb{Q}_p$ is $\{x\}_p = 0$ if $\gamma(x) \geqslant 0$, and it is

$$\{x\}_p = p^{\gamma}(x_0 + x_1 p + \ldots + x_{-\gamma-1} p^{-\gamma-1}) \text{ if } \gamma(x) \leqslant -1. \qquad (1.3)$$

Denote by $\mathbb{Q}_p^{\times 2}$ the multiplicative *group of squares* of p-adic numbers.

In order a number $x \in \mathbb{Q}_p^{\times}$ belongs to $\mathbb{Q}_p^{\times 2}$, it is necessary and safficient that $\gamma(x)$ is even and

$$\left(\frac{x_0}{p}\right) = 1, p \neq 2; \quad x_1 = x_2 = 0, p = 2.$$

Here

$$\left(\frac{a}{p}\right), \quad a \in Z, a \not\equiv 0 (\operatorname{mod} p)$$

is the *Legendre symbol* which equal to 1 or -1 subject to if the number x is quadratic residue or non-residue module p.

Thus the group $\mathbb{Q}_p^{\times}/\mathbb{Q}_p^{\times 2}$ consists of four elements $(1, \epsilon, p, \epsilon p)$ where ϵ is any unit of the field $\mathbb{Q}_p$ which is not a square in $\mathbb{Q}_p$ if $p \neq 2$, and it consists of eight elements $\{1, 2, 3, 5, 6, 7, 10, 14\}$ if $p = 2$.

§2. Some Functions on $\mathbb{Q}_p$

Characters of the field $\mathbb{Q}_p$. Let $\chi(x)$ be an *additive character* of the field $\mathbb{Q}_p$,

$$\chi(x + y) = \chi(x)\chi(y), \quad |\chi(x)| = 1, x, y \in \mathbb{Q}_p. \qquad (2.1)$$

Standard additive character of the field $\mathbb{Q}_p$ has the form

$$\chi_p(x) = \exp(2\pi i \{x\}_p) \qquad (2.2)$$

where $\{x\}_p$ is the rational part of $x \in \mathbb{Q}_p$ which is defined by the formula (1.3).

The general form of an additive character of the field $\mathbb{Q}_p$ is

$$\chi(x) = \chi_p(\xi x) = \exp(2\pi i \{\xi x\}_p) \qquad (2.3)$$

for some $\xi \in \mathbb{Q}_p$.

Let $\pi(x)$ be *multiplicative character* of the field $\mathbb{Q}_p$,

$$\pi(xy) = \pi(x)\pi(y), \quad |\pi(x)| = 1, x, y \in \mathbb{Q}_p^{\times}. \qquad (2.4)$$

The general form of a multiplicative character of the field $\mathbb{Q}_p$ is

$$\pi(x) = \pi_{i\alpha,\theta}(x) = |x|_p^{i\alpha}\theta(x), \quad x \in \mathbb{Q}_p^\times \tag{2.5}$$

where $\alpha \in \mathbb{R}$ is defined by the equality $\pi(p) = p^{-i\alpha}$ and $\theta(t), t \in Z_p^\times$ is a character of the compact group $Z_p^\times$ normalized by the condition $\theta(p) = 1$. (The set of the latters is countable and discrete.)

If the unitarity condition $|\pi(x)| = 1$ in (2.4) is not fulfilled then the function $\pi(x)$ is a *representation* of the group $\mathbb{Q}_p^\times$ in $\mathbb{C}$, and its general form is given by the formula (2.5) in which $i\alpha$ is any complex number, so that

$$\pi_{\alpha,\theta}(x) = |x|_p^{\alpha-1}\theta(x), \quad x \in \mathbb{Q}_p^\times, \alpha \in \mathbb{C}. \tag{2.5$'$}$$

Such functions are called *quasi-characters*. A quasi-character $\pi(x) = |x|_p^{\alpha-1}$ for which $\theta = 1$ is called *principal quasi-character.*

Let $d \notin \mathbb{Q}_p^{\times 2}$. Without loss of generality it is possible to suppose that d is square free of p-adic numbers, that is it is one of the listed in §1 forms, $p, \epsilon, p\epsilon, |\epsilon|_p = 1, \epsilon \notin \mathbb{Q}_p^{\times 2}$ for $p \neq 2$, and $2, 3, 5, 6, 7, 10{,}14$ for $p = 2$.

Denote by $\mathbb{Q}_p^\times(d)$ the set of p-adic numbers in $\mathbb{Q}_p^\times$ which are representable in the form $\alpha^2 - d\beta^2$, $\alpha, \beta \in \mathbb{Q}_p$; $\mathbb{Q}_p^\times(d)$ is a multiplicative group.

The *Hilbert symbol* $\left(\frac{x,y}{p}\right), x, y \in \mathbb{Q}_p^\times$ by definition is equal to 1 or -1 subject to the form $x\alpha^2 + y\beta^2 - \gamma^2$ represents nontrivially zero in $\mathbb{Q}_p$ or not.

The Hilbert symbol has the following obvious properties [5]:

$$\left(\frac{x,y}{p}\right) = \left(\frac{y,x}{p}\right), \quad \left(\frac{x,-x}{p}\right) = 1, \quad \left(\frac{x,yz}{p}\right) = \left(\frac{x,y}{p}\right)\left(\frac{x,z}{p}\right),$$

and besides

$$\left(\frac{p,\epsilon}{p}\right) = \left(\frac{\epsilon_0}{p}\right), \quad \left(\frac{\epsilon,\eta}{p}\right) = 1, \quad p \neq 2;$$

$$\left(\frac{2,\epsilon}{2}\right) = (-1)^{(\epsilon^2-1)/2}, \quad \left(\frac{\epsilon,\eta}{2}\right) = (-1)^{(\epsilon-1)(\eta-1)/4}, \quad p = 2.$$

Here ϵ and η are any units of the field $\mathbb{Q}_p$.

From here it follows a criterion in order that a p-adic number x belongs to $\mathbb{Q}_p^\times(d)$ for $p \neq 2$. *In order that $x \in \mathbb{Q}_p^\times(d)$ it is necessary and sufficient: for $d = \epsilon$ $\gamma(x)$ is even; for $d = p$ $\gamma(x)$ is even and $\left(\frac{x_0}{p}\right) = 1$ or $\gamma(x)$ is odd*

and $\left(\frac{-x_0}{p}\right) = 1$*; for* $d = p\epsilon$ $\gamma(x)$ *is even and* $\left(\frac{x_0}{p}\right) = 1$ *or* $\gamma(x)$ *is odd and* $\left(\frac{-x_0}{p}\right) = -1$. (Similar criterion takes place and for $p = 2$.)

Hence, *The group* $\mathbb{Q}_p^\times/\mathbb{Q}_p^\times(d)$ *is isomorphic to the group* $(1, -1)$*, and the function*

$$\mathrm{sgn}_{p,d}x = \begin{cases} 1, & x \in \mathbb{Q}_p^\times(d), \\ -1, & x \notin \mathbb{Q}_p^\times(d) \end{cases} \tag{2.6}$$

is a multiplicative character of the group $\mathbb{Q}_p^\times$.

Directly from the definitions it follows

$$\mathrm{sgn}_{p,d}x = \Big(\frac{x, -dx}{p}\Big), x \in \mathbb{Q}_p^\times, \quad d \notin \mathbb{Q}_p^{\times 2}.$$

(Note that always $\left(\frac{x,-dx}{p}\right) = 1$ if $d \in \mathbb{Q}_p^{\times 2}$.)

λ_p*-function of field* $\mathbb{Q}_p$ is defined by the following way [1a)],[6a)]

$$\lambda_p(x) = \begin{cases} 1, \quad \gamma(x) = 2k, \quad p \neq 2, \\ \sqrt{\left(\frac{-1}{p}\right)}\left(\frac{x_0}{p}\right), \quad \gamma(x) = 2k+1, \quad p \neq 2, \\ \exp[\pi i(1/4 + x_1)], \quad \gamma(x) = 2k, \quad p = 2, \\ \exp[\pi i(1/4 + x_1/2 + x_2)], \quad \gamma(x) = 2k+1, \quad p = 2. \end{cases}$$

Properties of λ_p-function $\mathbb{Q}_p^\times \to \mathbb{C}$.

$$|\lambda_p(x)| = 1, \quad \lambda_p(x)\lambda_p(-x) = 1;$$

$$\lambda_p(x) = \lambda_p(y), \quad xy \in \mathbb{Q}_p^{\times 2};$$

$$\frac{\lambda_p(x)\lambda_p(y)}{\lambda_p(x+y)} = \lambda_p\Big(\frac{xy}{x+y}\Big);$$

$$\lambda_p(x)\lambda_p(y) = \Big(\frac{x, y}{p}\Big)\lambda_p(xy)\lambda_p(1). \tag{2.7}$$

Puting in (2.7) $y = -dx$ and using the formula (2.6) we obtain relation [6a)]

$$\mathrm{sgn}_{p,d}x = \lambda_p(x)\lambda_p(-dx)\lambda_p(d)\lambda_p(-1), x \in \mathbb{Q}_p^\times, \quad d \notin \mathbb{Q}_p^{\times 2}. \tag{2.8}$$

Note the following formulae [6a)]

$$\mathrm{sgn}_{p,d} x = \begin{cases} \left(\frac{x_0}{p}\right)^{\gamma(d)} \left(\frac{d_0}{p}\right)^{\gamma(x)} \left(\frac{-1}{p}\right)^{\gamma(x)\gamma(d)}, & p \neq 2, \\ (-1)^{d_1 x_1 + (d_1+d_2)\gamma(x) + (x_1+x_2)\gamma(d)}, & p = 2. \end{cases} \tag{2.9}$$

In particular, for $d \equiv 3 (\bmod 4)$ we have [6b)]

$$\mathrm{sgn}_{p,d} x = \begin{cases} 1, \quad \left(\frac{d}{p}\right) = 1, \\ (-1)^{\gamma(x)}, \quad \left(\frac{d}{p}\right) = -1, p \neq 2, p \neq d, \\ \left(\frac{d}{p}\right)(-1)^{\gamma}(x), \quad p = d, \\ (-1)^{x_1}, \quad p = 2, d \equiv 7 (\bmod 8), \\ (-1)^{x_1+\gamma(x)}, \quad p = 2, d \equiv 3 (\bmod 8). \end{cases}$$

Note the following infinite products, which are valid for $x, y \in \mathbb{Q}^\times$

$$|x|_\infty \prod_{p=2}^{\infty} |x|_p = 1, \quad |x|_\infty = |x|; \tag{2.10}$$

$$\chi_\infty(x) \prod_{p=2}^{\infty} \chi_p(x) = 1, \quad \chi_\infty(x) = \exp(-2\pi i x); \tag{2.11}$$

$$\lambda_\infty(x) \prod_{p=2}^{\infty} \lambda_p(x) = 1, \quad \lambda_\infty(x) = \exp(-i\pi/4 \,\mathrm{sgn}\, x); \tag{2.12}$$

$$\left(\frac{x, y}{\infty}\right) \prod_{p=2}^{\infty} \left(\frac{x, y}{p}\right) = 1 \tag{2.13}$$

where $x, y \in \mathbb{Q}_p^\times$ and

$$\left(\frac{x, y}{\infty}\right) = \begin{cases} -1, \quad x < 0, y < 0, \\ 1, \ \text{otherwise} \ ; \end{cases}$$

$$\mathrm{sgn}_{\infty,d} x \prod_{p=2}^{\infty} \mathrm{sgn}_{p,d} x = 1 \tag{2.14}$$

where

$$\operatorname{sgn}_{\infty,d} x = \begin{cases} \operatorname{sgn} x, & d < 0, \\ 1, & d > 0. \end{cases}$$

Infinite products in formulas (2.10)–(2.14) converge for all rational x and y as only finite number of factors in them are different from 1. Formulas of such kind are called *adelic*.

Denote: $\Omega(|x|_p)$ is the characteristic function of disk B_0, so $\Omega(t) = 1$, if $0 \leqslant t \leqslant 1$ and $\Omega(t) = 0$, if $t > 1$; $\delta(|x|_p - p^\gamma)$ is the characteristic function of circumference S_γ; $\delta(x_\ell - k)$ is the characteristic function of the set $[x \in \mathbb{Q}_p : x_\ell = k]$, $k = 1, 2, \dots, p-1$ for $\ell = 0$ and $k = 0, 1, \dots, p-1$ for $\ell = 1, 2, \dots$.

§3. Analytic Functions

Let $\mathscr{O}$ be an open set in $\mathbb{Q}_p$. Function $f : \mathscr{O} \to \mathbb{Q}_p$ is called *analytic* in $\mathscr{O}$ if for any point $a \in \mathscr{O}$ there exists a $\gamma \in Z$ such that in the disk $B_\gamma(a) \subset \mathscr{O}$ it is represented by a convergent power series

$$f(x) = \sum_{k=o}^{\infty} c_k (x-a)^k. \tag{3.1}$$

Radius of convergence $r = r(f)$ of the series (3.1) is

$$r = p^\sigma, \quad \sigma = -\frac{1}{\ln p} \overline{\lim}_{k\to\infty} \frac{1}{k} \ln |f_k|_p.$$

The series (3.1) converges if, and only if, the series

$$\sum_{k=o}^{\infty} |c_k| p^{\gamma k}$$

converges, and it is possible to differentiate it term by term in $B_\gamma(a)$ infinite numbers of times,

$$f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)\dots(k-n+1) c_k (x-a)^{k-n}, \quad n = 1, 2, \dots, \tag{3.2}$$

and also

$$c_k = \frac{f^{(k)}(a)}{k!}, \quad k = 0, 1, \dots . \tag{3.3}$$

By every differentiation of series (3.1) the radius of convergence of the differentiated series (3.2) may only increase.

The functions e^x, $\ln x$, $\sin x$, $\cos x$, $\operatorname{tg} x$, $\arcsin x$, $\operatorname{arctg} x$ are analytic, they are defined by the following series

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}, \quad x \in G_p, \tag{3.4}$$

$$\ln x = \ln[1-(1-x)], \quad x \in J_p; \quad \ln x = -\sum_{k=1}^{\infty} \frac{x^k}{k}, \quad x \in G_p, \tag{3.5}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad x \in G_p, \tag{3.6}$$

$$\cos x = \sum_{k=0} \frac{(-1)^k}{(2k)!} x^{2k}, \quad x \in G_p, \tag{3.7}$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x}, \quad x \in G_p, \tag{3.8}$$

$$\arcsin x = \sum_{k=o}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2(2k+1)} x^{2k+1}, \quad x \in G_p, \tag{3.9}$$

$$\operatorname{arctg} x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad x \in G_p. \tag{3.10}$$

The following relations are valid

$$(e^x)' = e^x, \quad e^x e^y = e^{x+y}, \quad x, y \in G_p, \tag{3.11}$$

$$|e^x|_p = 1, \quad |e^x - 1|_p = |x|_p, \quad x \in G_p, \tag{3.12}$$

$$\ln(xy) = \ln x + \ln y, \quad x, y \in G_p, \tag{3.13}$$

$$|\ln(1+x)|_p = |x|_p, \quad x \in G_p, \tag{3.14}$$

$$\ln e^x = x, \quad x \in G_p; \quad e^{\ln x} = x, \quad x \in J_p. \tag{3.15}$$

The function e^x realizes the analytic diffeomorphism of additive group G_p onto multiplicative group J_p. The invers map is realized by the function $\ln x$.

All formulas of classical trigonometry are valid. Their proofs easily follow from the formal relation

$$e^{ix} = \cos x + i \sin x, \quad x \in G_p \tag{3.16}$$

where the symbol e^{ix} is defined by series (3.3) provided that $i^2 = -1$. In particular,

$$\sin^2 x + \cos^2 x = 1, \quad x \in G_p. \tag{3.17}$$

$$e^{\theta x} = \cos x + \theta \sin x, \quad x \in G_p, \quad \theta^2 = -1, \theta \in \mathbb{Q}_p \tag{3.18}$$

(the last is possible only for $p \equiv 1 (\mathrm{mod}\, 4)$).

Functions $\sin x$ and $\operatorname{tg} x$ realize the analytic isomorphysm of group G_p onto G_p; invers maps are given by functions $\arcsin x$ and $\operatorname{arctg} x$ respect.

§4. The Haar Measure on $\mathbb{Q}_p$.

As $\mathbb{Q}_p$ is a commutative group on addition so on it there exists an ivariant measure (uniqe up to a factor), the Haar measure, which we denote by $d_p x$,

$$d_p(x + a) = d_p x, a \in \mathbb{Q}_p; \quad d_p(ax) = |a|_p d_p x, a \in \mathbb{Q}_p^\times.$$

Normalize the measure $d_p x$ by the condition

$$\int_{Z_p} d_p x = 1. \tag{4.1}$$

The normed Haar measure $d_p^\times x$ on $\mathbb{Q}_p^\times$ is

$$d_p^\times x = (1 - p^{-1})^{-1} \frac{d_p x}{|x|_p}, \quad d_p^\times (ax) = d_p^\times x, a, x \in \mathbb{Q}_p^\times \tag{4.2}$$

so

$$\int_{Z_p^\times} d_p^\times x = 1.$$

Let $M \subset \mathbb{Q}_p$ be a measurable set (on the Haar measure). Integral of a function $f : M \to \mathbb{C}$ on the set M we will write in the form

$$\int_M f(x) d_p x, \quad \int f(x) d_p x = \int_{\mathbb{Q}_p} f(x) d_p x.$$

Let $1 \leqslant q \leqslant \infty$ be. The set of functions $f : \mathbb{Q}_p \to \mathbb{C}$ for which $f(x) = 0, x \notin M$ and

$$\|f\|_q = \Big[\int_M |f(x)|^q d_p x\Big]^{1/q} < \infty, \text{ if } q < \infty,$$

$$\|f\|_\infty = \operatorname*{vraisup}_{x \in M} |f(x)| < \infty, \text{ if } q = \infty,$$

we denote by $\mathscr{L}^q(M)$, $\quad \mathscr{L}^q = \mathscr{L}^q(\mathbb{Q}_p)$. If $\mathscr{O}$ is an open set in $\mathbb{Q}_p$ then the set of functions $f : \mathscr{O} \to \mathbb{C}$ for which for any compact $K \subset \mathscr{O} \quad f \in \mathscr{L}^q(K)$ we denote by $\mathscr{L}^q_{\text{loc}}(\mathscr{O})$, $\mathscr{L}^q_{\text{loc}} = \mathscr{L}^q_{\text{loc}}(\mathbb{Q}_p)$.

Functions of the set $\mathscr{L}^1_{\text{loc}}(\mathscr{O})$ are called *locally-integrable* in $\mathscr{O}$.

Let a function f be in $\mathscr{L}^1_{\text{loc}}(\mathbb{Q}_p^\times)$. *(Improper) integral* of a function f on $\mathbb{Q}_p$,

$$\int f(x) d_p x = \sum_{\gamma=-\infty}^{\infty} \int_{S_\gamma} f(x) d_p x,$$

is called the limit (if it exists)

$$\lim_{N,M\to\infty} \int_{B_N \setminus B_{-M-1}} = \lim_{N,M\to\infty} \sum_{\gamma=-M}^{N} \int_{S_\gamma} f(x) dx.$$

Example. Integral

$$\int_{Z_p} |x|^{\alpha-1} d_p x = \frac{1-p^{-1}}{1-p^{-\alpha}} \tag{4.3}$$

exists for $\operatorname{Re}\alpha > 0$.

The formula of change of variables in integral: *if $x(y)$ is an analytic diffeomorphism of a clopen set $D' \subset \mathbb{Q}_p$ onto $D \subset \mathbb{Q}_p$, and also $x'(y) \neq 0, y \in D'$, then for any $f \in \mathscr{L}^1(D)$ the formula is valid*

$$\int_D f(x) d_p x = \int_{D'} f(x(y)) |x'(y)|_p d_p y. \tag{4.4}$$

Example. Let $x = (py)^{-1}, d_p x = p|y|_p^{-2} d_p y$ be. Then owing to (4.3) we have

$$\int_{|x|_p > 1} |x|_p^{\alpha-1} d_p x = p^\alpha \int_{Z_p} |y|_p^{-\alpha-1} d_p y = \frac{1-p^{-1}}{p^{-\alpha}-1}, \operatorname{Re}\alpha < 0.$$

Example. The linear-fractional transformation is

$$x = \frac{ay+b}{cy+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(\mathbb{Q}_p, 2),$$

$$d_p x = \frac{|ad-bc|_p}{|cx+d|_p^2} d_p y.$$

§5. n-Dimensional Space $\mathbb{Q}_p^n$

Space $\mathbb{Q}_p^n = \mathbb{Q}_p \times \mathbb{Q}_p \times \ldots \times \mathbb{Q}_p$ (n times) consists of points $x = (x_1, x_2, \ldots, x_n), x_j \in \mathbb{Q}_p, j = 1, 2, \ldots, n$ supplied with the norm

$$|x|_p = \max_{1 \leqslant j \leqslant n} |x_j|_p. \tag{5.1}$$

This norm possesses properties 1)–3) §1 so the space $\mathbb{Q}_p^n$ is ultrametric (non-Archimedian).

Scalar product

$$(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n, \quad x, y \in \mathbb{Q}_p^n$$

satisfies the inequality

$$|(x, y)|_p \leqslant |x|_p |y|_p, \quad x, y \in \mathbb{Q}_p^n.$$

We denote the Haar measure on $\mathbb{Q}_p^n$ by $d_p^n x = d_p x_1 d_p x_2 \ldots d_p x_n$, $d_p x_1 = d_p x$,

$$d_p^n(x+a) = d_p^n x, a \in \mathbb{Q}_p^n, \quad d_p^n(Ax) = |\det A|_p d_p^n x$$

where $x \to Ax$ is a linear isomorphism of $\mathbb{Q}_p^n$ onto $\mathbb{Q}_p^n$ ($\det A \neq 0$).

Henceforth we agree in integrals on whole space $\mathbb{Q}_p^n$ to omit a domain of inegration,

$$\int_{\mathbb{Q}_p^n} f(x) d_p^n x = \int f(x) d_p^n x.$$

Spaces of functions $\mathscr{L}^q(M)$ and $\mathscr{L}^q_{loc}(\mathscr{O})$, $M, \mathscr{O} \in \mathbb{Q}_p^n$ are defined analogously to the case $n = 1$ (see §4).

As in the case $n = 1$ with the help of the notions introduced we define: $B_\gamma^n(a)$ is the ball of radius p^γ with the center at point $a = (a_1, a_2, \ldots, a_n) \in$

$\mathbb{Q}_p^n$ and $S_\gamma^n(a)$ is the sphere of radius p^γ with the center at point a; $B_\gamma^n(0) = B_\gamma^n$, $B_\gamma^1(a) = B_\gamma(a)$, $S_\gamma^n(o) = S_\gamma^n$, $S_\gamma^1(a) = S_\gamma(a)$,

$$B_\gamma^n(a) = B_\gamma(a_1) \times B_\gamma(a_2) \times \ldots \times B_\gamma(a_n).$$

The Fubini theorem. *If a function $f : \mathbb{Q}_p^{n+m} \to \mathbb{C}$ is such that the repeated integral*

$$\int \Big[\int |f(x,y)| d_p^m y \Big] d_p^n x$$

exists then f is in $\mathscr{L}^1(\mathbb{Q}_p^{n+m})$ and the aqualities are valid

$$\int \Big[\int f(x,y) d_p^m y \Big] d_p^n x = \int f(x,y) d_p^n x d_p^m y = \int \Big[\int f(x,y) d_p^n x \Big] d_p^m y. \tag{5.2}$$

Change of variables. *It $x = x(y)$ is an analytic diffeomorphism of a clopen set $D' \subset \mathbb{Q}_p^n$ onto set $D \subset \mathbb{Q}_p^n$ and also*

$$\det \frac{\partial x(y)}{\partial y} = \det \left(\frac{\partial x_k}{\partial y_j} \right) \neq 0, y \in D'$$

then for any $f \in \mathscr{L}^1(D)$ the equality is valid

$$\int_D f(x) d_p^n x = \int_{D'} f(x(y)) |\det \frac{\partial x(y)}{\partial y}|_p d_p^n y. \tag{5.3}$$

The Lebesgue theorem on passage to the limit under the sign of integral. *If a sequence $f_k, k \to \infty$ of functions $f_k \in \mathscr{L}^1$ converges almost everywhere to a function $f(x)$ and there exists a function $\psi \in \mathscr{L}^1$ such that*

$$|f_k(x)| \leqslant \psi(x) \text{ for almost every } x \in \mathbb{Q}_p^n$$

then the equality is valid

$$\lim_{k \to \infty} \int f_k(x) d_p^n x = \int f(x) d_p^n x.$$

§6. Generalized Functions on $\mathbb{Q}_p^n$

Let $\mathscr{O}$ be an open set in $\mathbb{Q}_p^n$. A function $\varphi : \mathscr{O} \to \mathbb{C}$ is called *locally-constant in* $\mathscr{O}$ if for any point $x \in \mathscr{O}$ there exists $\gamma \in Z$ such that

$$\varphi(x + x') = \varphi(x), x' \in B_\gamma^n, \quad x \in \mathscr{O}.$$

The set of all locally-constant functions in $\mathscr{O}$ we denote by $\mathscr{E}(\mathscr{O})$; $\mathscr{E} = \mathscr{E}(\mathbb{Q}_p^n)$. Every functions in $\varphi \in \mathscr{E}(\mathscr{O})$ is continuous on $\mathscr{O}$. Its support, which is the closure of points $x \in \mathscr{O}$ for which $\varphi(x) \neq 0$, we will denote by $\operatorname{spt} \varphi$.

Examples.

$$|x|_p \in \mathscr{E}(\mathbb{Q}_p^n \backslash \{0\}),$$

$$\chi_p((\xi, x)) \in \mathscr{E}, \quad \xi \in \mathbb{Q}_p^n.$$

A function $\varphi \in \mathscr{E}(\mathscr{O})$ is called *test function in* $\mathscr{O}$ (the Bruhat-Schwartz function) if its support is compact in $\mathscr{O}$. The set of test functions in $\mathscr{O}$ we denote by $\mathscr{S}(\mathscr{O})$; $\mathscr{S} = \mathscr{S}(\mathbb{Q}_p^n)$. Every function in $\mathscr{S}(\mathscr{O})$ is uniformly locally-constant in $\mathscr{O}$.

Examples.

$$\Omega_k(x) = \Omega(p^{-k}|x|_p) \in \mathscr{S}, \quad k \in Z, \tag{6.1}$$

$$\Delta_k(x) = p^k \Omega(p^k |x|_p) \in \mathscr{S}, \quad k \in Z. \tag{6.2}$$

$$|x|_p \Omega(|x|_p) \in \mathscr{S}(\mathbb{Q}_p^n \backslash \{0\}).$$

$$\chi_p((\xi, x))\Omega(|x|_p) \in \mathscr{S}, \quad \xi \in \mathbb{Q}_p^n.$$

$$\delta(|x|_p - p^\gamma) \in \mathscr{S}(S_\gamma), \quad \gamma \in Z.$$

$$\delta(|x|_p - p^\gamma)\delta(x_0 - k) \in \mathscr{S}(S_\gamma), k = 1, 2, \dots, p-1, \quad \gamma \in Z.$$

If K is an open compact in $\mathbb{Q}_p^n$ then θ_K is in $\mathscr{S}(K)$. Here θ_M is the characteristic function of a set $M \subset \mathbb{Q}_p^n : \theta_M(x) = 1, x \in M$, $\theta_M(x) = 0, x \notin M$.

Convergence in $\mathscr{S}(\mathscr{O})$,

$$\varphi_k \to 0, k \to \infty \;\; \mathscr{S}(\mathscr{O}),$$

means:

(i) there exists a compact $K \subset \mathscr{O}$ not depending on k such that $\operatorname{spt} \varphi_k \subset K$;

(ii) there exists $\gamma \in Z$ depending neither k nor x such that

$$\varphi_k(x + x') = \varphi_k(x), x' \in B_\gamma^n, \quad x \in K;$$

$(iii)\varphi_k(x) \Rightarrow 0, x \in K, k \to \infty$.

Generalized function on $\mathscr{O}$ is called any linear continuous functional $f : \varphi \to (f, \varphi)$ on $\mathscr{S}(\mathscr{O})$. The set of all generalized functions on $\mathscr{O}$ we denote by $\mathscr{S}'(\mathscr{O})$; $\mathscr{S}' = \mathscr{S}'(\mathbb{Q}_p^n)$.

Convergence in $\mathscr{S}'(\mathscr{O})$,

$$f_k \to 0, k \to \infty \text{ in } \mathscr{S}'(\mathscr{O}),$$

is defined as the weak convergence of functionals in $\mathscr{S}'(\mathscr{O})$, that is

$$(f_k, \varphi) \to 0, k \to \infty, \quad \varphi \in \mathscr{S}(\mathscr{O}).$$

Every linear on $\mathscr{S}(\mathscr{O})$ *functional* f *is continuous on* $\mathscr{S}(\mathscr{O})$, that is $f \in \mathscr{S}'(\mathscr{O})$.

In an open set $\mathscr{O}$ there exists *"decomposition of unity"* with functions in $\mathscr{S}(\mathscr{O})$, namely if

$$\mathscr{O} = \cup_{k=1}^{\infty} G_k, \quad G_k \cap G_j = \emptyset, k \neq j$$

where $G_k, k = 1, 2, \ldots$ are clopen compacts, so the equality holds

$$\sum_{k=1}^{\infty} \theta_{G_k}(x) = 1, \quad x \in \mathscr{O}. \tag{6.3}$$

A generalized function f in $\mathscr{S}'(\mathscr{O})$ *vanishes in an open set* $\mathscr{O}' \subset \mathscr{O}$ if $(f, \varphi) = 0, \varphi \in \mathscr{S}(\mathscr{O}')$, besides we write: $f(x) = 0, x \in \mathscr{O}'$. Generalized functions f and g in $\mathscr{S}'(\mathscr{O})$ *coincide in* (equal in) $\mathscr{O}' \subset \mathscr{O}$, $f = g$ in $\mathscr{O}'$, iff $f(x) - g(x) = 0$ for $x \in \mathscr{O}'$. The largest open set in which vanishes $f \in \mathscr{S}'(\mathscr{O})$ is called *null-set* of f, and it is denoted by $\mathscr{O}_f \subset \mathscr{O}$. A closed in $\mathscr{O}$ set $\mathscr{O} \backslash \mathscr{O}_f$ is called *support of* f, and it is denoted by $\operatorname{spt} f$, $\operatorname{spt} f = \mathscr{O} \backslash \mathscr{O}_f$.

We denote the set of generalized functions with compact support in $\mathscr{O}$ by $\mathscr{E}'(\mathscr{O})$, $\mathscr{E}' = \mathscr{E}'(\mathbb{Q}_p^n)$; $\mathscr{E}'(\mathscr{O})$ is the strongly conjugate space to $\mathscr{E}(\mathscr{O})$.

Example. *δ-Function*

$$(\delta, \varphi) = \varphi(0), \quad \operatorname{spt} \delta = \{0\}. \tag{6.4}$$

Conversely, every $f \in \mathscr{S}', \operatorname{spt} f = \{0\}$ has the form

$$f = C\delta \tag{6.5}$$

where $C \neq 0$ is an arbitrary constant.

A sequence $\{\delta_k, k \to \infty\}$ of functions $\delta_k(x)$ in $\mathscr{S}$ is called $\delta-$*like* if it is bounded in $\mathscr{L}^1$ and for any $\gamma \in Z$ the limit relation holds

$$\int_{B_\gamma^n} \delta_k(x) d_p^n x \to 1, \quad \int_{\mathbb{Q}_p^n \setminus B_\gamma^n} |\delta_k(x)| d_p^n x \to 0, \quad k \to \infty.$$

Thus,

$$\delta_k \to \delta, k \to \infty \text{ in } \mathscr{S}'. \tag{6.6}$$

A sequence $\{\omega_k, k \to \infty\}$ of functions $\omega_k(x)$ in $\mathscr{S}$ is called *1-like* if it is the Fourier-transform (see below §7) of some $\delta-$like sequence $\{\delta_k, k \to \infty\}$.

1-Like sequence is bounded in $\mathscr{L}^\infty$, and for any$\gamma \in Z$

$$\omega_k(x) \rightrightarrows 1, x \in B_\gamma^n, k \to \infty.$$

Thus,

$$\omega_k \to 1, k \to \infty \text{ in } \mathscr{S}'. \tag{6.7}$$

If $f \in \mathscr{L}^1_{\text{loc}}(\mathscr{O})$ so $f \in \mathscr{S}'(\mathscr{O})$, besides

$$(f, \varphi) = \int f(x)\varphi(x) d_p^n x, \quad \varphi \in \mathscr{S}(\mathscr{O}). \tag{6.8}$$

Generalized functions of the form (6.8) are called *regular* in $\mathscr{O}$; the others are called *singular.* δ-Function is singular in $\mathbb{Q}_p^n$, and it is regular in $\mathbb{Q}_p^n \setminus \{0\}$.

Let 0 be in $\mathscr{O}$. If $f \in \mathscr{S}'(\mathscr{O} \setminus \{0\})$ then it admit an *extension (regularization)* $f_1 \in \mathscr{S}'(\mathscr{O})$ on $\mathscr{O}$ and all its regularizations, $\operatorname{reg} f$, are given by the formula

$$\operatorname{reg} f = f_1 + C\delta, \tag{6.9}$$

where C is an arbitrary constant and f_1 can be choosen in the form

$$(f_1, \varphi) = (f, \varphi - \Omega_\gamma \varphi(0)), \quad \varphi \in \mathscr{S}(\mathscr{O}),$$

besides $\gamma \in Z$ is such that $B_\gamma^n \subset \mathscr{O}$. Note, that this fact does not take place for generalized functions of real arguments! As an example of such f is function $f(x) = \exp x^{-1}$.

For $f = |x|_p^{-1}$ as a regularization it is possible to take the functional

$$(\operatorname{reg} |x|_p^{-1}, \varphi) = \int |x|_p^{-1} [\varphi(x) - \Omega(|x|_p)\varphi(0)] d_p x, \quad \varphi \in \mathscr{S}.$$

The generalized function $\operatorname{reg}|x|_p^{-1}$ gives another example of singular generalized function on $\mathbb{Q}_p^n$.

The product of a generalized function $f \in \mathscr{S}'(\mathscr{O})$ on a function $a \in \mathscr{E}(\mathscr{O})$ is defined by the formula

$$(af, \varphi) = (f, a\varphi), \varphi \in \mathscr{S}(\mathscr{O}), \quad af \in \mathscr{S}'(\mathscr{O}). \tag{6.10}$$

Examples.

$$a(x)\delta(x) = a(0)\delta(x).$$

If $f \in \mathscr{L}^1_{loc}(\mathscr{O})$, so af coincides with ussual product of functions $a(x)$ and $f(x)$.

If $f \in \mathscr{S}'(\mathscr{O})$ and $\operatorname{spt} f$ is clopen set in $\mathscr{O}$, so

$$f(x) = \theta_{\operatorname{spt} f}(x) f(x). \tag{6.11}$$

Finelly, if $f \in \mathscr{S}'$, so

$$\omega_k f \to f, k \to \infty \text{ in } \mathscr{S}' \tag{6.12}$$

where $\{\omega_k, k \to \infty\}$ is any 1-like sequence.

In $\mathscr{S}'(\mathscr{O})$ *theorem on "piecewise sewing"* is valid. Let a collection of generalized functions $f_k \in \mathscr{S}'(G_k), k = 1, 2, \dots$ be given where $G_k, k = 1, 2, \dots$ are clopen compacts satisfying conditions $G_k \cap G_j = \emptyset, k \neq j$. Then there exists a (unique) generalized function $f \in \mathscr{S}'(\mathscr{O})$ such that $f = f_k$ in $G_k, k = 1, 2, \dots$ and $\mathscr{O} = \cup_{k \leqslant 1} G_k$.

Therem on "nucleus". *Let $\varphi \to A(\varphi)$ be a linear map of $\mathscr{S}(\mathscr{O}), \mathscr{O} \in \mathbb{Q}_p^n$ into $\mathscr{S}'(\mathscr{O}'), \mathscr{O}' \in \mathbb{Q}_p^m$. Then there exists a (unique) generalized function $f \in \mathscr{S}'(\mathscr{O} \times \mathscr{O}')$ such that*

$$\Big(A(\varphi), \psi\Big) = (f, \varphi(x)\psi(y)), \varphi \in \mathscr{S}(\mathscr{O}), \psi \in \mathscr{S}(\mathscr{O}').$$

The spaces $\mathscr{S}(\mathscr{O})$ and $\mathscr{S}'(\mathscr{O})$ are complete, reflexive and nuclear; $\mathscr{S}(\mathscr{O})$ is dense in $\mathscr{S}'(\mathscr{O})$.

Linear change of variables $y = Ax + b, \quad \det A \neq 0$, maps a generalized function $f(y)$ in $\mathscr{S}'(\mathscr{O}')$ in the generalized function $f(Ax + b)$ in $\mathscr{S}'(\mathscr{O})$ by the formula

$$\Big(f(Ax + b), \varphi\Big) = \frac{1}{|\det A|_p} \Big(f(y), \varphi(A^{-1}(y - b))\Big), \quad \varphi \in \mathscr{S}(\mathscr{O}). \tag{6.13}$$

Examples. $\delta(x) = \delta(-x), \quad (\delta(x - x_0), \varphi) = \varphi(x_0)$.

The direct product $f(x)\times g(y)$ of generalized functions $f\in\mathscr{S}'(\mathscr{O}_1)$, $\mathscr{O}_1\subset\mathbb{Q}_p^n$ and $g\in\mathscr{S}'(\mathscr{O}_2)$, $\mathscr{O}_2\subset\mathbb{Q}_p^m$ is defined by the formula

$$(f(x)\times g(y),\varphi)=\big(f(x),(g(y),\varphi(x,y))\big),\quad \varphi\in\mathscr{S}(\mathscr{O}_1\times\mathscr{O}_2).$$

The direct product is *commutative*, so

$$f(x)\times g(y)=g(y)\times f(x)\in\mathscr{S}'(\mathscr{O}_1\times\mathscr{O}_2). \tag{6.14}$$

For $g=1$ the formula (6.14) takes the form

$$\Big(f(x),\int_{\mathscr{O}_2}\varphi(x,y)d^my\Big)=\int_{\mathscr{O}_2}\big(f(x),\varphi(x,y)\big)d^my,\quad f\in\mathscr{S}'(\mathscr{O}_1),$$

$$\varphi\in\mathscr{S}(\mathscr{O}_1\times\mathscr{O}_2) \tag{6.15}$$

(generalization of the Fubini theorem, see §5).

Convolution $f*g$ of generalized functions $f\in\mathscr{E},\operatorname{spt}f\in B_N^n$ and $g\in\mathscr{S}'$ is defined by the equality

$$(f*g,\varphi)=(f(x)\times g(y),\Omega_N(x)\varphi(x+y)),\quad \varphi\in\mathscr{S}. \tag{6.16}$$

On the base of this definition the convolution of generalized functions f and g in $\mathscr{S}'$ is defined by

$$(f*g,\varphi)=\lim_{k\to\infty}(f(x)\times g(y),\Omega_k(x)\varphi(x+y)=\lim_{k\to\infty}((\Omega_kf)*g,\varphi)$$

if the limit exists for any $\varphi\in\mathscr{S}$, so $f*g\in\mathscr{S}'$.

If the convolution $f*g$ exists then the convolution $g*f$ also exists and they both are equal (*commutativity of convolution*),

$$f*g=g*f. \tag{6.17}$$

Examples.

$$f*\delta=\delta*f=f,\quad f\in\mathscr{S}'. \tag{6.18}$$

If $f\in\mathscr{S}'$ and $\psi\in\mathscr{S}$ then the convolution $f*\psi$ is a locally-constant function in $\mathbb{Q}_p^n$, besides

$$(f*\psi)(x)=(f(y),\psi(x-y)),x\in\mathbb{Q}_p^n. \tag{6.19}$$

If $\{\delta_k, k\to\infty\}$ is a $\delta-$like sequence then

$$f*\delta_k \to f, k\to\infty \text{ in } \mathscr{S}', \quad f\in\mathscr{S}'. \tag{6.20}$$

If f, g in $\mathscr{L}^1_{\mathrm{loc}}$ and there exists a function $q\in\mathscr{L}^1_{\mathrm{loc}}$ such that

$$\int_{B_k} f(x-y)g(y)d^n_p y \to q(x), k\to\infty \text{ in } \mathscr{S}'$$

then

$$f*g = q(x). \tag{6.21}$$

If $f\in\mathscr{S}'$ and the convoluton $f*1$ exists then it is a constant. We call this constant *integral* of generalized function f on the whole space $\mathbb{Q}^n_p$, and we denote it by

$$G\!\!\int f(x)d^n_p x = f*1. \tag{6.22}$$

This definition is equivalent to the following:

$$G\!\!\int f(x)d^n_p x = \lim_{k\to\infty}(f,\Omega_k) \tag{6.23}$$

if the limit exists.

If $f\in\mathscr{S}'$ and $\operatorname{spt} f\subset D$ where D is a clopen set in $\mathbb{Q}^n_p$, so $f=\theta_D f$, and the integral (6.22) we denote by

$$G\!\!\int_D f(x)d^n_p x.$$

In particular, if $f\in\mathscr{S}'$, $\varphi\in\mathscr{S}$ and $\operatorname{spt}\varphi\subset B_\gamma$, so

$$G\!\!\int_{B_\gamma} f(x)\varphi(x)d^n_p x = (f,\varphi). \tag{6.24}$$

If $f\in\mathscr{S}', \operatorname{spt} f\subset B_\gamma$, so

$$G\!\!\int_{B_\gamma} f(x)d^n_p x = (f,\Omega_\gamma). \tag{6.25}$$

The notion of integral of a generalized function introduced is in fact an extension of the notion of integral on the Haar measure (see §§1,4).

Example.

$$G\!\!\int \delta(x)d_px = 1.$$

Multipliation of generalized functions. Let f, g be in $\mathscr{S}'$. We call *product* $f \cdot g$ the functional defined by the equality

$$f \cdot g = \lim_{k\to\infty} (f * \Delta_k)g$$

if the limit exists in $\mathscr{S}'$, so $f \cdot g \in \mathscr{S}'$.

If the product $f \cdot g$ exists, so the product $g \cdot f$ also exists and they are equal (*commutativity of product*)

$$f \cdot g = g \cdot f. \tag{6.26}$$

Examples.

$$a \cdot f = af, \quad a \in \mathscr{E}, f \in \mathscr{S}'.$$

In particular,

$$f \cdot 1 = 1 \cdot f = f, \quad f \in \mathscr{S}',$$

$$a(x) \cdot \delta(x) = a(0)\delta(x)$$

if a is a continuous function in a vicinity of 0,

$$|x|_p^\alpha \cdot \delta(x) = 0, \alpha > 0, \quad |x|_p \cdot \operatorname{reg}|x|_p^{-1} = 1. \tag{6.27}$$

As in the case of real field, a question arises: is it possible to define the product of any generalized functions by such a way that it was associative and commutative? The answer is negative. Well-known example by L. Schwartz in p-adic case seems so. If such product would exist so owing to (6.27) we would have the following contradictory chain of equalities:

$$0 = 0\cdot\operatorname{reg}|x|_p^{-1} = (|x|_p\cdot\delta(x))\cdot\operatorname{reg}|x|_p^{-1} = \delta(x)\cdot(|x|_p\cdot\operatorname{reg}|x|_p^{-1}) = \delta(x)\cdot 1 = \delta(x).$$

§7. The Fourier Transform

Let φ be in $\mathscr{S}$. The *Fourier transform* $\tilde{\varphi} = F[\varphi]$ is defined by the formula

$$\tilde{\varphi}(\xi) = \int \varphi(x)\chi_p((\xi, x))d_p^n x, \quad x \in \mathbb{Q}_p^n.$$

The Fourier transform is a linear isomorphism of $\mathscr{S}$ onto $\mathscr{S}$ and the *inversion formula* for the Fourier transform is valid

$$\varphi(x) = \int \tilde{\varphi}(\xi)\chi_p(-(x,\xi))d_p^n\xi, \quad \varphi \in \mathscr{S}.$$

Examples.

$$\tilde{\Omega}_k = \Delta_k, \quad \tilde{\Delta}_k = \Omega_k, \quad k \in Z. \tag{7.1}$$

The Fourier transform $\tilde{f} = F[f]$ of a generalized function $f \in \mathscr{S}'$ is defined by the formula

$$(\tilde{f}, \varphi) = (f, \tilde{\varphi}), \quad \varphi \in \mathscr{S},$$

so that $\tilde{f} \in \mathscr{S}'$.

The Fourier transform $f \to \tilde{f}$ is a linear isomorphism of $\mathscr{S}'$ onto $\mathscr{S}'$ and the inversion formula is valid

$$f = F^{-1}[\tilde{f}] = F[\check{\tilde{f}}], \quad f \in \mathscr{S}'$$

where $\check{f}(x) = f(-x)$.

Examples.

$$\tilde{\delta} = 1, \quad \tilde{1} = \delta; \tag{7.2}$$

$$F[f(Ax+b)] = |\det A|_p^{-1}\chi_p\big(-(A^{-1}b,\xi)\big)F[f(A^{-1}\xi)], \quad \det A \neq 0. \tag{7.3}$$

In particular,

$$F[f(x-b)] = \chi_p((b,\xi))F[f(\xi)]; \tag{7.4}$$

$$\tilde{\check{f}} = \check{\tilde{f}}. \tag{7.5}$$

If $f \in \mathscr{L}^1$ then

$$\tilde{f}(\xi) = \int f(x)\chi_p((\xi,x))d^nx, \tag{7.6}$$

and also $\tilde{f}$ is continuous in $\mathbb{Q}_p^n$ and $\tilde{f}(\xi) \to 0$, $|\xi|_p \to \infty$ (analogy of the *Riemann-Lebesgue theorem*).

If $f \in \mathscr{L}^1_{\text{loc}}$ and there exists a function $q \in \mathscr{L}^1_{\text{loc}}$ such that

$$\int_{B_k^n} f(x)\chi_p((\xi,x))d_p^nx \to q(\xi), k \to \infty \text{ in } \mathscr{S}'$$

then

$$\tilde{f} = q. \tag{7.7}$$

If $f \in \mathscr{S}', \operatorname{spt} f \subset B_\gamma^n$ then

$$\tilde{f}(\xi) = \big(f(x), \Omega_\gamma(x)\chi_p((\xi, x))\big). \tag{7.8}$$

If $f \in \mathscr{L}^2$ then

$$\int_{B_k^n} f(x)\chi_p((\xi, x))d_p^n x \to \tilde{f}(\xi), \quad k \to \infty \text{ in } \mathscr{L}^2. \tag{7.9}$$

The operator $f \to \tilde{f}$ is unitary in $\mathscr{L}^2$ so the *Parseval-Steklov equality* is valid

$$\|f\| = \|\tilde{f}\|, \quad f \in \mathscr{L}^2 \tag{7.10}$$

where the norm $\|f\| = \|f\|_2 = (f, f)^{1/2}$ is defined in §4 and the *scalar product* (f, g) in $\mathscr{L}^2$ is equal to

$$(f, g) = \int f(x)\bar{g}(x)d_p^n x, \quad f, g \in \mathscr{L}^2.$$

The *Cauchy-Buniakowski inequality* is valid

$$|(f, g)| \leqslant \|f\|\|g\|, \quad f, g \in \mathscr{L}^2.$$

If $f \in \mathscr{L}^2$ then

$$\lim_{k\to\infty} p^{-k/2} \int_{B_k} |f(x)|d_p^n = 0. \tag{7.11}$$

Theorem. *Let f, g be in $\mathscr{S}'$. The convolution $f * g$ exists if, and only if, there exists the product $\tilde{f} \cdot \tilde{g}$ and the equalities are valid*

$$\widetilde{f * g} = \tilde{f} \cdot \tilde{g}, \quad \widetilde{f \cdot g} = \tilde{f} * \tilde{g}. \tag{7.12}$$

Note the following useful formula

$$\int_{S_\gamma^n} \chi_p((x, \xi))d_p^n x = (1 - p^{-n})p^{\gamma n}\Omega(p^\gamma|\xi|_p) - q^{(k-1)n}\delta(|\xi|_p - p^{1-\gamma}) \tag{7.13}$$

whence

$$\int_{B_\gamma^n} \chi_p((x, \xi))d_p^n x = p^{\gamma n}\Omega(p^\gamma|\xi|_p). \tag{7.14}$$

The *Gaussian integral* $G_p(a;\xi)$ is called the Fourier transform of the function $\chi_p(ax^2), a \in \mathbb{Q}_p^\times, p=\infty,2,3,5,\ldots$,

$$G_p(a,\xi)=\int \chi_p(ax^2+\xi x)d_px=\lambda_p(a)|2a|_p^{-1/2}\chi_p\big(-\xi^2/4a\big). \tag{7.15}$$

The following adelic formula is valid

$$G_\infty(a;\xi)\prod_{p=2}^{\infty}G_p(a;\xi)=1, \quad a\in\mathbb{Q}^\times,\xi\in\mathbb{Q} \tag{7.16}$$

which follows from the adelic formulae (2.10)–(2.12).

§8. Homogeneous Generalized Functions

Let $\pi(x)=\pi_{\alpha,\theta}(x)=|x|_p^{\alpha-1}\theta(x)$ be a quasi-character of the field $\mathbb{Q}_p$ (see (2.5′)). A generalized function $f\in\mathscr{S}'$ is called *homogeneous* with respect to a quasi-character $\pi_{\alpha,\theta}$ if

$$f(tx)=\pi_{\alpha,\theta}(t)f(x), t\in\mathbb{Q}_p^\times, \quad x\in\mathbb{Q}_p^\times. \tag{8.1}$$

Homogeneous generalized functions with respect to a principal quasi-character

$$\pi_{\alpha,1}(x)=|x|_p^{\alpha-1}$$

are called homogeneous of degree $\alpha-1$.

A quasi-character $\pi_{\alpha,\theta}(x)$ defines a homogeneous with respect to itself generalized function $\pi_{\alpha,\theta}$ by the formula

$$(\pi_{\alpha,\theta},\varphi)=\int |x|_p^{\alpha-1}\theta(x)\varphi(x)d_px, \quad \varphi\in\mathscr{S}. \tag{8.2}$$

The generalized function $\pi_{\alpha,\theta}$ *for* $\theta\neq 1$ *is entire on* α*; for* $\theta=1$ *it is holomorphic on* α *everywhere except simple poles*

$$\alpha_k=2k\pi i/\ln p, \quad k\in Z$$

with residue $\frac{1-p^{-1}}{\ln p}\delta(x)$.

Note that the generalized function $|x|_p^{\alpha-1}$ defined in domain $\operatorname{Re}\alpha>0$ by the formula (8.2) is analytically continued from this domain to the domain $\operatorname{Re}\alpha\leqslant 0, \quad \alpha\neq\alpha_k, k\in Z$ by the formula

$$(|x|_p^{\alpha-1},\varphi)=(1-p^{-\alpha})^{-1}\int |x|_p^{\alpha-1}[\varphi(x)-\varphi(x/p)]d_px$$

$$= \int |x|_p^{\alpha-1}[\varphi(x) - \varphi(0)]d_px, \quad \varphi \in \mathscr{S} \tag{8.3}$$

as

$$\int |x|_p^{\alpha-1} = 0, \quad \alpha \neq \alpha_k, k \in Z.$$

For $\alpha = \alpha_k, k \in Z$ the quasi-character $\pi_{0,1}(x) = |x|_p^{-1}$ corresponds the generalized function $\delta(x)$ of degree -1; conversely, every homogeneous generalized function $f \in \mathscr{S}'$ of degree -1 has the form $f(x) = C\delta(x)$ where C is some constant.

The Fourier transform of $\pi_{\alpha,\theta}$ is a homogeneous generalized function $\tilde{\pi}_{\alpha,\theta}$ with respect to the quasi-character

$$\pi_{\alpha,\theta}^{-1}(\xi)|\xi|_p^{-1} = |\xi|_p^{-\alpha}\bar{\theta}(\xi) = \pi_{1-\alpha,\bar{\theta}}(\xi), \tag{8.4}$$

so

$$\tilde{\pi}_{\alpha,\theta} = \Gamma_p(\pi_{\alpha,\theta})\pi_{1-\alpha,\bar{\theta}}. \tag{8.5}$$

Here $\Gamma_p(\pi_{\alpha,\theta})$ is *gamma-function* of field $\mathbb{Q}_p$ for quasi-character $\pi_{\alpha,\theta}(x)$,

$$\Gamma_p(\pi_{\alpha,\theta}) = \tilde{\pi}_{\alpha,\theta}(1) = \int |x|_p^{\alpha-1}\theta(x)\chi_p(x)d_px. \tag{8.6}$$

In particular, for $\theta = 1$, if we denote

$$\Gamma_p(\alpha) = \Gamma_p(|x|_p^{\alpha-1}),$$

we get for the gamma-function $\Gamma_p(\alpha)$ of a principal quasi-character $|x|_p^{\alpha-1}$ the representation

$$\Gamma_p(\alpha) = \int |x|_p^{\alpha-1}\chi_p(x)d_px = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, \quad \alpha \neq \alpha_k, k \in Z. \tag{8.7}$$

For $\epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1, p \neq 2$

$$\theta(x) = \mathrm{sgn}_{p,\epsilon}x = |x|_p^{\pi i/\ln p} = (-1)^{\gamma(x)}$$

we denote

$$\tilde{\Gamma}_p(\alpha) = \Gamma_p(|x|_p^{\alpha-1}\mathrm{sgn}_{p,\epsilon}x).$$

For $\tilde{\Gamma}_p$-function from (8.7) it follows the expression

$$\tilde{\Gamma}_p(\alpha) = \Gamma_p(\alpha + \pi i/\ln p) = \frac{1+p^{\alpha-1}}{1+p^{-\alpha}}, \quad \alpha \neq \alpha_k - \pi i/\ln p, k \in Z. \tag{8.8}$$

Note the particular formulae for gamma-function when d=-1 (cf. §2),

$$\Gamma_p(\mathrm{sgn}_{p,-1}x|x|^{\alpha-1}) = \begin{cases} \Gamma_p(\alpha) = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}, p \equiv 1 (\mathrm{mod}\, 4), \\ \tilde{\Gamma}_p(\alpha) = \frac{1+p^{\alpha-1}}{1+p^{-\alpha}}, p \equiv 3 (\mathrm{mod}\, 4), \\ 2i4^{\alpha-1}, p = 2. \end{cases}$$

The following equality is valid

$$\Gamma_p(\pi_{\alpha,\theta})\Gamma_p(\pi_{1-\alpha,\bar{\theta}}) = \theta(-1). \tag{8.9}$$

In particular,

$$\Gamma_p(\alpha)\Gamma_p(1-\alpha) = 1. \tag{8.10}$$

Convolution of homogeneous generalized function $\pi_{\alpha,\theta}$ and $\pi_{\beta,\theta'}$ exists and it is a homogeneous generalized function with respect to quasi-character

$$\pi_{\alpha,\theta}(x)\pi_{\beta,\theta'}(x)|x|_p^{-1} = \pi_{\alpha+\beta,\theta\theta'}(x),$$

and thus

$$\pi_{\alpha,\theta} * \pi_{\beta,\theta'} = B_p(\pi_{\alpha,\theta}, \pi_{\beta,\theta'})\pi_{\alpha+\beta,\theta\theta'}. \tag{8.11}$$

Here $B_p(\pi_{\alpha,\theta}, \pi_{\beta,\theta'})$ is *beta-function* of field $\mathbb{Q}_p$ for quasi-characters $\pi_{\alpha,\theta}$ and $\pi_{\beta,\theta'}$,

$$B_p(\pi_{\alpha,\theta}, \pi_{\beta,\theta'}) = \big(\pi_{\alpha,\theta} * \pi_{\beta,\theta'}\big)(1) = \frac{\Gamma_p(\pi_{\alpha,\theta})\Gamma_p(\pi_{\beta,\theta'})}{\Gamma_p(\pi_{\alpha+\beta,\theta\theta'})}$$

$$= \Gamma_p(\pi_{\alpha,\theta})\Gamma_p(\pi_{\beta,\theta'})\Gamma_p(\pi_{\gamma,\theta''})\theta''(-1), \quad \alpha+\beta+\gamma = 1, \theta\theta'\theta'' = 1. \tag{8.12}$$

In particular, for principal quasi-characters ($\theta = \theta' = 1$) formula (8.12) takes the form

$$B_p(\alpha, \beta) = \Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma), \quad \alpha+\beta+\gamma = 1 \tag{8.13}$$

where is denoted

$$B_p(\alpha, \beta) = B_p(|x|_p^{\alpha-1}, |x|_p^{\beta-1}).$$

Note another symmetric expression for the beta-function $B_p(\alpha, \beta)$ [13]:

$$B_p(\alpha, \beta) = (1-p^{-1})\big[(1-p^{-\alpha})^{-1} + (1-p^{-\beta})^{-1} + (1-p^{-\gamma})^{-1} - 1\big] - 1,$$

$$\alpha+\beta+\gamma = 1. \tag{8.14}$$

By introducing the analogy of the Euler gamma- and beta-functions,

$$\gamma_p(\alpha) = \int_{Z_p} |x|_p^{\alpha-1}\chi_p(x)d_px = \frac{1-p^{-1}}{1-p^{-\alpha}},$$

$$b_p(\alpha,\beta) = \int_{Z_p} |x|_p^{\alpha-1}|1-x|_p^{\beta-1}d_px = \gamma_p(\alpha)+\gamma_p(\beta)-1,$$

we get the equality (8.14) in the form

$$B_p(\alpha,\beta) = \frac{1}{2}b_p(\alpha,\beta)+\frac{1}{2}b_p(\alpha,\gamma)+\frac{1}{2}b_p(\beta,\gamma)+\frac{1}{p}-\frac{1}{2}, \quad \alpha+\beta+\gamma=1. \tag{8.15}$$

Rank $\rho(\theta)$ of a quasi-character $\pi_{\alpha,\theta}$ (and a chracter θ) is called such integer number $k \geqslant 0$ that $\theta(t)=1$ for $|1-t|_p \leqslant p^{-k}, t \in Z_p^\times$ and $\theta(t) \neq 1$ for $|1-t|_p = p^{1-k}, t \in Z_p^\times$. It is clear that zero rank has only the principal quasi-character $|x|_p^\alpha$.

For quasi-characters of the rank $k \geqslant 1$ the following formulas are valid [4]:

$$\Gamma_p(\pi_{\alpha,\theta}) = p^{\alpha k}a_{p,k}(\theta), \tag{8.16}$$

$$a_{p,\gamma}(\theta) = \int_{S_0} \theta(t)\chi_p(p^{-\gamma}t)d_pt, \quad \gamma \geqslant 1, \tag{8.17}$$

$$a_{p,\gamma}(\theta) = 0, \gamma \neq k, \quad |a_{p,k}(\theta)| = p^{-k/2}, \tag{8.18}$$

$$a_{p,k}(\theta)a_{p,k}(\bar\theta) = p^{-k}\theta(-1), \tag{8.19}$$

$$\int_{S_k} \theta(p^kx)\chi_p(\xi x)d_px = p^ka_{p,k}(\theta)\bar\theta(\xi)\delta(|\xi|_p-1), \tag{8.20}$$

$$\Gamma_p(\pi_{\alpha,\theta})\Gamma_p(\pi_{\alpha,\theta}^{-1}) = p^k\theta(-1), \tag{8.21}$$

$$\Gamma_p(\pi_{\alpha+1,\theta}) = p^k\Gamma_p(\pi_{\alpha,\theta}). \tag{8.22}$$

Example. The rank of a quasi-character

$$\pi_{\alpha,\theta}(x) = |x|_p^{\alpha-1}\mathrm{sgn}_{p,d}x, \quad |d|_p = 1/p, p \neq 2 \tag{8.23}$$

is equal 1. Therefore and owing to (8.16) and (8.19)

$$\Gamma_p(\pi_{\alpha,\theta}) = \pm p^{\alpha-1/2}\sqrt{\mathrm{sgn}_{p,d}(-1)}. \tag{8.24}$$

The operator (8.2)

$$\varphi \to (\pi_{\alpha,\theta}, \varphi) \equiv M^{\pi}[\varphi]$$

is called *the Mellin transform* of function $\varphi \in \mathscr{S}$ with respect to a quasi-character $\pi_{\alpha,\theta}(x)$. For $\theta = 1$ the function $M^{|x|_p^{\alpha-1}}[\varphi] \equiv M^{\alpha}[\varphi]$ is called simply the Mellin transform of a function $\varphi \in \mathscr{S}$. Owing to (8.3) it can be represented in the following form

$$M^{\alpha}[\varphi] = (1 - p^{-\alpha})^{-1} \int |x|_p^{\alpha-1}[\varphi(x) - \varphi(x/p)] d_p x, \quad \alpha \neq \alpha_k, k \in Z.$$

According to (8.2) and (8.5) the equality takes place

$$M^{\pi}[\tilde{\varphi}] = \Gamma_p(\pi_{\alpha,\theta}) M^{\tilde{\pi}}[\varphi] \quad \varphi \in \mathscr{S}. \tag{8.25}$$

For $\theta = 1$ formula (8.25) takes the form

$$M^{\alpha}[\tilde{\varphi}] = \Gamma_p(\alpha) M^{1-\alpha}[\varphi]. \tag{8.25$'$}$$

The Mellin transform of $Z_p^{\times}$-invariant (generalized) functions and its inversion. A function $\varphi \in \mathscr{S}(\mathbb{Q}_p^{\times})$ is called $Z_p^{\times}$-*invariant* if $\varphi(x) = \varphi(t|x|_p)$, $t \in Z_p^{\times}$, $x \in \mathbb{Q}_p^{\times}$ or, in other words,

$$\varphi(x) = (1 - p^{-1})^{-1} \int_{Z_p} \varphi(t|x|_p) d_p t \equiv S[\varphi](|x|_p).$$

Every $Z_p^{\times}$-invariant function $\varphi \in \mathscr{S}(\mathbb{Q}_p^{\times})$ is represented uniquely in the form

$$\varphi(x) = \sum_{\gamma} \varphi_{\gamma} \delta(|x|_p - p^{\gamma}), \quad \varphi_{\gamma} = \varphi(p^{\gamma}) = S[\varphi](p^{\gamma}).$$

Thus the subspace of space $\mathscr{S}(\mathbb{Q}_p^{\times})$ consisting of $Z_p^{\times}$-invariant functions is isomorphic to the space of finite sequences $\{\varphi_{\gamma}, \gamma \in N\}$ where N is a bounded subset of Z.

A generalized function $f \in \mathscr{S}(\mathbb{Q}_p^{\times})$ is called $Z_p^{\times}$- invariant if

$$(f, \varphi) = (f(x), S[\varphi](|x|_p)), \quad \varphi \in \mathscr{S}(\mathbb{Q}_p^{\times}).$$

Any $Z_p^\times$-invariant generalized function $\varphi \in \mathscr{S}'(\mathbb{Q}_p^\times)$ is represented uniquely in the form

$$f(x) = \sum_\gamma f_\gamma \delta(|x|_p - p^\gamma), \quad f_\gamma = (1-p^{-1})^{-1} p^{-\gamma}(f(x), \delta(|x|_p - p^\gamma)),$$

so a subspace of the space $\mathscr{S}'(\mathbb{Q}_p^\times)$ consisting of $Z_p^\times$-invariant generalized functions is isomorphic to the space sequences $\{f_\gamma, \gamma \in Z\}$.

If $\varphi \in \mathscr{S}(\mathbb{Q}_p^\times)$ is $Z_p^\times$-invariant function then its Mellin transform

$$M^\alpha[\varphi] = \int |x|^{\alpha-1} S[\varphi](|x|_p) d_p x = (1-p^{-1}) \sum_{\gamma \in M} \varphi_\gamma p^{\alpha\gamma}$$

is entire functin of α, and the invertion formula is valid [16]

$$\varphi(x) = \frac{\ln p}{2\pi i(1-p^{-1})} \int_{\sigma - i\pi/\ln p}^{\sigma + i\pi/\ln p} M^\alpha[\varphi] |x|_p^{-\alpha} d\alpha. \tag{8.26}$$

The formula (8.26) is extended also on $Z_p^\times$-invariant generalized functions f from $\mathscr{S}'(\mathbb{Q}_p^\times)$ satisfying the condition

$$\sum_{\gamma \in Z} |f_\gamma| p^{c\gamma} < \infty$$

for some c. Its Mellin transform

$$M^\alpha[f] = (f(x), |x|_p^{\alpha-1}) = (1-p^{-1}) \sum_{\gamma \in Z} f_\gamma p^{\gamma\alpha}$$

is a holomorphic function of α in half-plane $\operatorname{Re}\alpha < c$, and the invertion formula (8.26) is valid for f, and also integral (8.26) does not depends on $\sigma < c$.

Space $\mathbb{Q}_p^n$. We restrict ourself by the case of a principal quasi-character $|x|_p^\alpha$. The generalized function $|x|_p^{\alpha-n}$ is homogeneous of degree $\alpha - n$, holomorphic on α everywhere exept simple poles $\alpha_k = 2k\pi i/\ln p, k \in Z$ with residue $\frac{1-p^{-n}}{\ln p}\delta(x)$; the formula of the Fourier transform is valid [10]

$$\widetilde{|x|_p^{\alpha-n}} = \Gamma_p^{(n)}(\alpha) |\xi|_p^{-\alpha}, \quad \alpha \neq \alpha_k, k \in Z \tag{8.27}$$

where $\Gamma_p^{(n)}$ is the gamma-function of vector space $\mathbb{Q}_p^n$ $(\Gamma_p^{(1)} = \Gamma_p)$,

$$\Gamma_p^{(n)}(\alpha) = \int |x|_p^{\alpha-n}\chi_p(x_1)d_p^n x = \frac{1-p^{\alpha-n}}{1-p^{-\alpha}}, \quad \alpha \neq \alpha_k, k \in Z, \tag{8.28}$$

$$\Gamma_p^{(n)}(\alpha)\Gamma_p^{(n)}(n-\alpha) = 1, \tag{8.29}$$

$$\Gamma_p^{(n)}(\alpha) = (-1)^{n-1}p^{(n-1)(n/2-\alpha)}\prod_{k=1}^{n-1}\Gamma_p(\alpha-k). \tag{8.30}$$

Beta-function $B_p^{(n)}$ of space $\mathbb{Q}_p^n$ is defined similar to (8.11) $(B_p^{(1)} = B_p)$ by the equality

$$|x|_p^{\alpha-n} * |x|_p^{\beta-n} = B_p^{(n)}(\alpha,\beta)|x|_p^{\alpha+\beta-n}, \tag{8.31}$$

$$B_p^{(n)}(\alpha,\beta) = \Gamma_p^{(n)}(\alpha)\Gamma_p^{(n)}(\beta)\Gamma_p^{(n)}(\gamma),$$
$$\alpha+\beta+\gamma = n, (\alpha,\beta) \neq (\alpha_k,\beta_j), (k,j) \in Z^2. \tag{8.32}$$

Adelic formulae for gamma- and beta-functions. For gamma-functions the following adelic formula is valid [2e)]

$$\Gamma_\infty(\alpha)\,\mathrm{reg}\prod_{p=2}^{\infty}\Gamma_p(\alpha) = 1, \quad \alpha \neq 0,1 \tag{8.33}$$

where Γ_∞ is the gamma-function of field $\mathbb{R}$,

$$\Gamma_\infty(\alpha) = \int |x|_p^{\alpha-1}\exp(-2\pi i x)dx$$

$$= 2(2\pi)^{-\alpha}\Gamma(\alpha)\cos\frac{\pi\alpha}{2} = \frac{\zeta(1-\alpha)}{\zeta(\alpha)} \tag{8.34}$$

where Γ is the Euler gamma-function and ζ is the Riemann zeta-function,

$$\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha} = \prod_{p=2}^{\infty}(1-p^{-\alpha})^{-1}, \quad \mathrm{Re}\,\alpha > 1.$$

Regularization of the divergent product in (8.33) is defined by means of the formula

$$\prod_{p=2}^{P}\Gamma_p(\alpha)\,\mathrm{AC}\prod_{p=P_1}^{\infty}(1-p^{-\alpha})^{-1}$$

$$= \frac{\zeta(\alpha)}{\zeta(1-\alpha)} \,\mathrm{AC} \prod_{p=P_1}^{\infty} (1-p^{\alpha-1})^{-1} \quad P = \infty, 2, 3, 5, \ldots \tag{8.35}$$

which folows from Tate's formula. Here P_1 is the prime number following the prime P; $\mathrm{AC}\, f(\alpha)$ is the analytic continuation on α of function $f(\alpha)$ which is holomorphic in some domain of the complex plane of the variable α.

Passing on to the limit in (8.35) as $P \to \infty$ in half-plane $\mathrm{Re}\,\alpha < 0$, denoting

$$\mathrm{reg} \prod_{p=2}^{\infty} \Gamma_p(\alpha) = \lim_{P\to\infty} \prod_{p=2}^{P} \Gamma_p(\alpha) \,\mathrm{AC} \prod_{p=P_1}^{\infty} (1-p^{-\alpha})^{-1},$$

and using equality (8.34) we get adelic formula (8.33). For $\mathrm{Re}\,\alpha \leqslant 0$ the $\mathrm{reg}\prod \Gamma_p(\alpha)$ is defined from (8.33) as the analytic continuation on α.

The similar adelic formula is valid also for beta-functions:

$$B_\infty(\alpha,\beta)\,\mathrm{reg} \prod_{p=2}^{\infty} B_p(\alpha,\beta) = 1 \tag{8.36}$$

where

$$B_\infty(\alpha,\beta) = \Gamma_\infty(\alpha)\Gamma_\infty(\beta)\Gamma_\infty(\gamma), \quad \alpha+\beta+\gamma = 1 \tag{8.37}$$

is beta-function of the field $\mathbb{R}$, and according to (8.13)

$$\mathrm{reg} \prod_{p=2}^{\infty} B_p(\alpha,\beta) = \prod_{x=\alpha,\beta,\gamma} \mathrm{reg} \prod_{p=2}^{\infty} \Gamma_p(x). \tag{8.38}$$

Note others symmetric expressions for B_∞:

$$\begin{aligned} B_\infty(\alpha,\beta) &= B(\alpha,\beta) + B(\alpha,\gamma) + B(\beta,\gamma) \\ &= \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} + \frac{\Gamma(\alpha)\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} + \frac{\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\beta+\gamma)} \\ &= \frac{4}{\pi} \prod_{x=\alpha,\beta,\gamma} \Gamma(x) \cos\frac{\pi x}{2} = \prod_{x=\alpha,\beta,\gamma} \frac{\zeta(1-x)}{\zeta(x)}. \end{aligned} \tag{8.39}$$

Adelic formula for the Riemann zeta-function,

$$\zeta(\alpha) = \sum_{n=1}^{\infty} n^{-\alpha} = \prod_{p=2}^{\infty} (1-p^{-\alpha})^{-1}.$$

$\zeta(\alpha)$ satisfies relation

$$\pi^{-\alpha/2}\Gamma(\alpha/2)\zeta(\alpha)=\pi^{-(1-\alpha)/2}\Gamma((1-\alpha)/2)\zeta(1-\alpha). \tag{8.40}$$

Denote

$$\zeta_\infty(\alpha)=\int \exp^{-\pi x^2}|x|^{\alpha-1}dx=\pi^{-\alpha/2}\Gamma(\alpha/2), \tag{8.41}$$

$$\zeta_p(\alpha)=\frac{1}{1-p^{-1}}\int_{Z_p}|x|_p^{\alpha-1}d_px=(1-p^{-\alpha})^{-1}, \tag{8.42}$$

$$\zeta_A(\alpha)=\zeta_\infty(\alpha)\zeta(\alpha). \tag{8.43}$$

Then the following formulae are valid:

$$\zeta_A(\alpha)=\zeta_A(1-\alpha),(\text{ cf. } (8.40)\ ), \tag{8.44}$$

$$\Gamma_\infty(\alpha)=\frac{\zeta_\infty(\alpha)}{\zeta_\infty(1-\alpha)}=\frac{\zeta(\alpha)}{\zeta(1-\alpha)} \tag{8.45}$$

(cf. (8.34)),

$$\zeta_\infty(\alpha)\prod_{p=2}^{\infty}\zeta_p(\alpha)=\zeta_A(\alpha). \tag{8.46}$$

Formula (8.45) is the adelic formula for the Riemann zeta-function.

§9. Quadratic Extensions of the Field $\mathbb{Q}_p$

Let $d\notin\mathbb{Q}_p^{\times 2}$ be a p-dic number. *Quadratic extension* of the field $\mathbb{Q}_p$ is the field $\mathbb{Q}_p(\sqrt{d})=\mathbb{Q}_p+\sqrt{d}\mathbb{Q}_p$. Let us describe all non-isomorphic fields $\mathbb{Q}_p(\sqrt{d})$. According to what has been said in §1 it is sufficient to consider integer rational numbers d, free of squares, i.e. $d=\pm p_1p_2\dots p_n,\quad d\neq 1$, where $p_1,p_2,\dots,p_n$ are different prime numbers.

The following cases are possible:

$$p\neq 2,p_1,\dots,p_n,\quad \left(\frac{d}{p}\right)=1,\quad \mathbb{Q}_p(\sqrt{d})\sim\mathbb{Q}_p;$$

$$p\neq 2,p_1,\dots,p_n,\quad \left(\frac{d}{p}\right)=-1,\quad \mathbb{Q}_p(\sqrt{d})\sim\mathbb{Q}_p(\sqrt{\epsilon}),\epsilon\notin\mathbb{Q}_p^{\times 2},|\epsilon|_p=1;$$

$$p\neq 2,p=p_i,\quad \left(\frac{d/p_i}{p}\right)=1,\quad \mathbb{Q}_p(\sqrt{d})\sim\mathbb{Q}_p(\sqrt{p});$$

$$p \neq 2, p = p_i, \quad \left(\frac{d/p_i}{p}\right) = -1, \quad \mathbb{Q}_p(\sqrt{d}) \sim \mathbb{Q}_p(\sqrt{p\epsilon}), \epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1;$$

$$p = 2, \quad d \equiv 3,5,7 (\mathrm{mod}\, 8), \quad \mathbb{Q}_2(\sqrt{d}) \sim \mathbb{Q}_2(\sqrt{\epsilon}), \epsilon = 3,5,7 \text{ resp. };$$

$$p = 2, \quad d/2 \equiv 1,3,5,7 (\mathrm{mod}\, 8), \quad \mathbb{Q}_2(\sqrt{d}) \sim \mathbb{Q}_2(\sqrt{2\epsilon}), \epsilon = 1,3,5,7 \text{ resp. }.$$

Note that $\mathbb{Q}_p(\sqrt{d})$ is the closure of the field $\mathbb{Q}(\sqrt{d}) = \mathbb{Q} + \sqrt{d}\mathbb{Q}$ on metric $\sqrt{|z\bar{z}|_p}$ where $z = x + \sqrt{d}y, \bar{z} = x - \sqrt{d}y,\ z\bar{z} = x^2 - dy^2, \quad x, y \in \mathbb{Q}$.

The Haar mesure $d_p z$ of field $\mathbb{Q}_p(\sqrt{d})$ we choose in the form

$$d_p z = 1/\delta d_p x d_p y, \quad z = x + \sqrt{d}y, x, y \in \mathbb{Q}_p \tag{9.1}$$

where $\delta = \delta_{p,d} = 2$ if $p = 2, d \equiv 5 (\mathrm{mod}\, 8)$ and $\delta = 1$ otherwise. The mesure $d_p z$ is normalized by the condition (see [2d)])

$$\int_{B_0^2} d_p z = 1, \quad B_0^2 = [z \in \mathbb{Q}_p(\sqrt{d}) : |z\bar{z}|_p \leqslant 1]. \tag{9.2}$$

The following equality is valid

$$d_p(az) = |a\bar{a}|_p d_p z, \quad a \in \mathbb{Q}_p^{\times}(\sqrt{d}). \tag{9.3}$$

The quantity $|a\bar{a}|_p$ is called *the module of automorphism* $z \to az$ of field $\mathbb{Q}_p(\sqrt{d})$.

The maximal compact subring $Z_p(\sqrt{d})$ of field $\mathbb{Q}_p(\sqrt{d})$ is

$$Z_p(\sqrt{d}) = [z \in \mathbb{Q}_p(\sqrt{d}) : |z\bar{z}|_p \leqslant 1], \quad Z_p = B_0^2;$$

its multiplicative subgroup is

$$Z_p^{\times}(\sqrt{d}) = [z \in \mathbb{Q}_p(\sqrt{d}) : |z\bar{z}|_p = 1];$$

its maximal ideal is

$$I_p(\sqrt{d}) = [z \in \mathbb{Q}_p(\sqrt{d}) : |z\bar{z}|_p < 1].$$

Residue classes $Z_p(\sqrt{d})/I_p(\sqrt{d})$ form the finite field of characteristic p called *residue field*; a number of its elements $q = q_{p,d}$ (is equal to p or p^2) is called *the module of field* $\mathbb{Q}_p(\sqrt{d})$. For special cases we have: for $p = 2, d \equiv$

$5 (\bmod 8)$, $q=4$ the residue field is $\{0, 1, 1/2 \pm \sqrt{5}/2\}$; for $d \not\equiv 5 (\bmod 8)$, $q=2$ the residue field is $\{0,1\}$; for $p \neq 2, |d|_p = 1$, $q = p^2$ the residue field is $\{k + \sqrt{d}j, k, j = 0, 1, \dots, p-1\}$; for $|d|_p = 1/p$, $q = p$ the residue field is $\{0, 1, \dots, p-1.\}$

The Fourier transform $\tilde{\varphi}(\zeta), \zeta = \xi + \sqrt{d}\eta$ of a test function $\varphi(z) \equiv \varphi(x,y)$ in $\mathscr{S}(\mathbb{Q}_p(\sqrt{d})) \sim \mathscr{S}(\mathbb{Q}_p^2)$ we define by the following formula

$$\tilde{\varphi}(\zeta) = \delta \sqrt{|4d|_p} \int \varphi(z) \chi_p(z\zeta + \tilde{z}\tilde{\zeta}) d_p z$$

$$= \sqrt{|4d|_p} \int \varphi(x,y) \chi_p(2x\xi + 2dy\eta) d_p x d_p y.$$

The invers Fourier transform is expressed by the equality

$$\varphi(z) = \delta \sqrt{|4d|_p} \int \tilde{\varphi}(\zeta) \xi_p(-z\zeta - \tilde{z}\tilde{\zeta}) d_p \zeta.$$

Thus the mesure $\delta\sqrt{|4d|_p} d_p z$ is self-dual with respect to the charater $\chi_p(z + \bar{z})$.

The generalized function

$$|z\bar{z}|_p^{\alpha-1} = |x^2 - dy^2|_p^{\alpha-1}$$

is defined by the equality (see §8)

$$(|z\bar{z}|_p^{\alpha-1}, \varphi) = \int_{|z\bar{z}|_p \leqslant 1} |z\bar{z}|_p^{\alpha-1} [\varphi(z) - \varphi(0)] d_p z$$

$$+ \int_{|z\bar{z}|_p > 1} |z\bar{z}|_p^{\alpha-1} d_p z + \varphi(0) \frac{1-q^{-1}}{1-q^{-\alpha}}, \quad \varphi \in \mathscr{S}(\mathbb{Q}_p(\sqrt{d}))$$

or, equivalently,

$$(|z\bar{z}|_p^{\alpha-1}, \varphi) = \int [|z\bar{z}|_p^{\alpha-1} [\varphi(z) - \varphi(0)] d_p z, \quad \varphi \in \mathscr{S}(\mathbb{Q}_p(\sqrt{d})).$$

Here we used formulas:

$$\int_{B_0^2} |z\bar{z}|_p^{\alpha-1} d_p z = \frac{1-q^{-1}}{1-q^{-\alpha}}, \quad \alpha \neq \alpha_k, k \in Z, \tag{9.4}$$

$$\int |z\bar{z}|_p^{\alpha-1} d_p z = 0, \quad \alpha \neq \alpha_k, k \in Z \tag{9.5}$$

where

$$\alpha_k = 2k\pi i/\ln q, \quad k \in Z. \tag{9.6}$$

The generalized function $|z\bar{z}|_p^{\alpha-1}$ (degree of homogeneity $2\alpha - 2$) is holomorphic on α everywhere exept simple poles $\alpha = \alpha_k, k \in Z$ (see (9.5)) with the residue $\frac{q-1}{q \ln q}\delta(x, y)$.

The Fourier transform formula is valid [2d)]

$$F[|z\bar{z}|_p^{\alpha-1}] = \Gamma_{p,d}(\alpha)|\zeta\bar{\zeta}|_p^{-\alpha}, \quad \alpha \neq \alpha_k, k \in Z \tag{9.7}$$

where

$$\Gamma_{p,d}(\alpha) = \delta\sqrt{|4d|_p}\int |z\bar{z}|_p^{\alpha-1}\xi_p(z+\bar{z})d_p z = \rho_{p,d}(\alpha)\Gamma_q(\alpha) \tag{9.8}$$

is gamma-function of the field $\mathbb{Q}_p(\sqrt{d})$;

$$\Gamma_q(\alpha) = \frac{1-q^{\alpha-1}}{1-q^{-\alpha}} \tag{9.9}$$

is *reduced gamma-function* of the field $\mathbb{Q}_p(\sqrt{d})$; and

$$\begin{aligned} \rho_{p,d}(\alpha) &= 1, \text{ if } |d|_p = 1, p \neq 2 \text{ or } d \equiv 5 (\operatorname{mod} 8), p = 2, \\ &= p^{\alpha-1/2}, \text{ if } |d|_p = 1/p, p \neq 2, \\ &= p^{2\alpha-1}, \text{ if } d \equiv 3 (\operatorname{mod} 4), p = 2, \\ &= p^{3\alpha-3/2}, \text{ if } |d|_2 = 1/2, p = 2. \end{aligned} \tag{9.10}$$

From (9.8)–(9.10) it follows the following relation for gamma-function of the field $\mathbb{Q}_p(\sqrt{d})$:

$$\Gamma_{p,d}(\alpha)\Gamma_{p,d}(1-\alpha) = 1. \tag{9.11}$$

Beta-function of the field $\mathbb{Q}_p(\sqrt{d})$ is introduced similar to §8. The convolution $|z\bar{z}|_p^{\alpha-1} * |z\bar{z}|_p^{\beta-1}$ exists for all complex (α, β) from the tube domain $\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}(\alpha+\beta) < 1$, and it is expressed by the integral

$$|z\bar{z}|_p^{\alpha-1} * |z\bar{z}|_p^{\beta-1} = \int |\zeta\bar{\zeta}|_p^{\alpha-1}|(z-\zeta)(\bar{z}-\bar{\zeta})|_p^{\beta-1} d_p\zeta$$

$$= B_{p,d}(\alpha,\beta)|z\bar{z}|_p^{\alpha+\beta-1} \tag{9.12}$$

where $B_{p,d}$ is beta-function of the field $\mathbb{Q}_p(\sqrt{d})$ [4]:

$$B_{p,d}(\alpha,\beta) = \int |\zeta\bar{\zeta}|_p^{\alpha-1}|(1-\zeta)(1-\bar{\zeta})|_p^{\beta-1} d_p\zeta$$

$$= \frac{\Gamma_{p,d}(\alpha)\Gamma_{p,d}(\beta)}{\delta\sqrt{|4d|_p}\Gamma_{p,d}(\alpha+\beta)}. \tag{9.13}$$

From equalities (9.8)–(9.13) it follows such symmetric expessions for beta-function:

$$B_{p,d}(\alpha,\beta) = \frac{1}{\delta\sqrt{|4d|_p}}\Gamma_{p,d}(\alpha)\Gamma_{p,d}(\beta)\Gamma_{p,d}(\gamma) = B_q(\alpha,\beta)$$

$$= \Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma), \quad \alpha+\beta+\gamma=1, (\alpha,\beta)\neq(\alpha_k,\beta_j), (k,j)\in Z^2. \tag{9.14}$$

Note that equalities (9.12)–(9.14) are valid for all (α,β) such that $(\alpha,\beta)\neq(\alpha_k,\alpha_j), (k,j)\in Z^2$.

We call *upper (lower) half-plain* of the field $\mathbb{Q}_p(\sqrt{d})$ a set of points $z = x+\sqrt{d}y$ for which $\mathrm{sgn}_{p,d}y = 1$ (resp. $\mathrm{sgn}_{p,d}y = -1$).

Generalized functions $(x\pm\sqrt{d}0)^{-1}$ are defined as the Fourier transform of functions

$$\theta_d^{\pm}(\xi) = \frac{1}{2}(1\pm\mathrm{sgn}_{p,d}\xi), \quad (x\pm\sqrt{d}0)^{-1} = \tilde{\theta}_d^{\pm}(x). \tag{9.15}$$

The following equalities are valid [4]

$$F[\theta_d^{\pm}](x) = (x\pm\sqrt{d}0)^{-1} = \frac{1}{2}\delta(x) + C_{p,d}\frac{\mathrm{sgn}_{p,d}x}{|x|_p}, \quad p\neq 2 \tag{9.16}$$

which are similar to the Sochozki formulae (for the field $\mathbb{R}$). Here a generalized function $\frac{\mathrm{sgn}_{p,d}x}{|x|_p}$ is defined by the equality

$$\left(\frac{\mathrm{sgn}_{p,d}x}{|x|_p},\varphi\right) = \int \frac{\mathrm{sgn}_{p,d}x}{|x|_p}\varphi(x)d_px, \quad \varphi\in\mathscr{S}, \tag{9.17}$$

$$C_{p,d} = \begin{cases} \sqrt{\frac{p}{p+1}}, \text{ if } |d|_p = 1, \\ \pm\frac{1}{2}\sqrt{p\mathrm{sgn}_{p,d}(-1)}, \quad \text{if } |d|_p = 1/p. \end{cases}$$

For Γ_q-function the following adelic formulae are valid [2d)]

$$\Gamma_\infty^2(\alpha)\,\mathrm{reg}\prod_{p=2}^{\infty}\Gamma_q^\nu(\alpha)=D^{1/2-\alpha},\quad d>0, \tag{9.18}$$

$$\Gamma_\omega(\alpha)\,\mathrm{reg}\prod_{p=2}^{\infty}\Gamma_q^\nu(\alpha)=|D|^{1/2-\alpha},\quad d<0 \tag{9.18$'$}$$

where Γ_∞ and Γ_ω are gamma-functions of fields $\mathbb{R}$ and $\mathbb{C}$ resp.;

$$\Gamma_\omega(\alpha)=2\int|z\bar{z}|^{\alpha-1}\exp(-4\pi ix)dxdy=(2\pi)^{1-2\alpha}\frac{\Gamma(\alpha)}{\Gamma(1-\alpha)};$$

$$=2(2\pi)^{-2\alpha}\Gamma^2(\alpha)\sin\pi\alpha=i\Gamma_\infty(\alpha)\tilde{\Gamma}(\alpha),$$

where

$$\tilde{\Gamma}(\alpha)=\int\mathrm{sgn}\,x|x|^{\alpha-1}\exp(-2\pi ix)dx=-2i(2\pi)^{-\alpha}\Gamma(\alpha)\sin\frac{\pi\alpha}{2};$$

$\nu=2$ if $d\in\mathbb{Q}_p^{\times 2}$ and $\nu=1$ if $d\notin\mathbb{Q}_p^{\times 2}$; D is the discriminant of the field $Q(\sqrt{d})$,

$$D=\begin{cases} d, & \text{if } d\equiv 1(\mathrm{mod}\,4)\\ 4d, & \text{if } d\equiv 2,3(\mathrm{mod}\,4)\end{cases}.$$

(We took the Haar mesure of field $\mathbb{C}$ in the form $|dz\wedge\bar{z}|=2dxdy$, $z=x+iy$.)

Regularization of the divergent infinite products in (9.18) is defined by the formula (cf. (8.35))

$$\prod_{p=2}^{P}\Gamma_q^\nu(\alpha)\,\mathrm{AC}\prod_{p=P_1}^{\infty}(1-q^{-\alpha})^{-\nu}=\frac{\zeta_d(\alpha)}{\zeta_d(1-\alpha)}\,\mathrm{AC}\prod_{p=P_1}^{\infty}(1-q^{-\alpha})^{-\nu},$$

$$P=\infty,2,3,5,\dots \tag{9.19}$$

which follows from general Tate's formula. Here ζ_d is Dedekind's zeta-function of the field $\mathbb{Q}(\sqrt{d})$,

$$\zeta_d(\alpha)=\prod_{p=2}^{\infty}(1-q^{-\alpha})^{-\nu},\quad \zeta_1(\alpha)=\zeta^2(\alpha).$$

The Dedekind zeta-function satisfies relation (cf. (8.40))

$$(2\pi)^{1-\alpha}\Gamma(\alpha)\zeta_d(\alpha) = (2\pi)^{\alpha}\Gamma(1-\alpha)\zeta_d(1-\alpha)|D|^{1/2-\alpha},$$

which is equivalent to relation (cf. (8.46))

$$\zeta_{A_d}(\alpha) = \zeta_{A_d}(1-\alpha)|D|^{1/2-\alpha},$$

where it is denoted

$$\zeta_{A_d}(\alpha) = (2\pi)^{1-\alpha}\Gamma(\alpha)\zeta_d(\alpha).$$

Passing on in (9.19) to the limit $P \to \infty$, denoting

$$\mathrm{reg}\prod_{p=2}^{\infty}\Gamma_q^{\nu}(\alpha) = \lim_{P\to\infty}\prod_{p=2}^{P}\Gamma_q^{\nu}(\alpha)\,\mathrm{AC}\prod_{p=P_1}^{\infty}(1-q^{-\alpha})^{-\nu}$$

and using equalities

$$\Gamma_{\infty}^{2}(\alpha) = D^{1/2-\alpha}\frac{\zeta_d(1-\alpha)}{\zeta_d(\alpha)}, \quad d>0, \tag{9.20}$$

$$\Gamma_{\omega}(\alpha) = |D|^{1/2-\alpha}\frac{\zeta_d(1-\alpha)}{\zeta_d(\alpha)}, \quad d<0 \tag{9.20$'$}$$

in the halp-plain $\mathrm{Re}\,\alpha < 0$ we obtain the adelic formulae (9.18). For remaining α $\mathrm{reg}\prod\Gamma_q^{-\nu}(\alpha)$ is defined from formulae (9.18) s analytic continuation on α.

Similar adelic formulae are valid also for beta-functions

$$B_{\infty}^{2}(\alpha,\beta)\,\mathrm{reg}\prod_{p=2}^{\infty}B_q^{\nu}(\alpha,\beta) = \sqrt{D}, \quad d>0, \tag{9.21}$$

$$B_{\omega}(\alpha,\beta)\,\mathrm{reg}\prod_{p=2}^{\infty}B_q^{\nu}(\alpha,\beta) = \sqrt{|D|}, \quad d<0 \tag{9.21$'$}$$

where B_{∞} and B_{ω} are beta-functions of fields $\mathbb{R}$ and $\mathbb{C}$ resp.,

$$B_{\omega}(\alpha,\beta) = \Gamma_{\omega}(\alpha)\Gamma_{\omega}(\beta)\Gamma_{\omega}(\gamma), \quad \alpha+\beta+\gamma=1 \tag{9.22}$$

and in accordance with the formula (9.14) (cf. (8.36))

$$\operatorname{reg}\prod_{p=2}^{\infty} B_q^{\nu}(\alpha,\beta) = \prod_{x=\alpha,\beta.\gamma} \operatorname{reg}\prod_{p=2}^{\infty}\Gamma_q^{\nu}(x).$$

Note another symmetric expressions for B_ω

$$B_\omega(\alpha,\beta) = 2\pi \prod_{x=\alpha,\beta,\gamma} \frac{\Gamma(x)}{\Gamma(1-x)} = \frac{2}{\pi^2}\prod_{x=\alpha,\beta,\gamma}\Gamma^2(x)\sin\pi x. \tag{9.23}$$

§10. The operator D^α

The generalized function

$$f_\alpha(x) = \frac{|x|_p^{\alpha-1}}{\Gamma_p(\alpha)}$$

is holomorphic on α everywhere exept simples poles $1+\alpha_k$, $\alpha_k = 2k\pi i/\ln p$, $k\in Z$ with the residue $\frac{1-p}{p\ln p}$, and also $f_{\alpha_k}=\delta$ and

$$f_\alpha * f_\beta = f_{\alpha+\beta}, \quad \alpha\neq 1+\alpha_k, \beta\neq 1+\alpha_j, \alpha+\beta\neq 1+\alpha_i, (k,j,i)\in Z^3.$$

Let $\alpha\in\mathbb{R}, \alpha\neq -1$ and $f\in\mathscr{S}'$ be such that the convolution $f_{-\alpha}*f$ exists in $\mathscr{S}'$. Operator $D^\alpha f = f_{-\alpha}*f$ is called for $\alpha>0$ the operator (fractional) *differentiation* of order α, nd for $\alpha<0$ the operator (fractional) *integration* of order $-\alpha$; for $\alpha=0$ $D^0 f=\delta * f = f$ is the identical operator [2a)].

Example. If $\alpha = 1$ $\varphi\in\mathscr{S}$ then

$$(D\varphi)(x) = \frac{p^2}{p+1}\int\frac{\varphi(x)-\varphi(y)}{|x-y|_p^2}d_py = \int|\xi|_p\tilde{\varphi}(\xi)\chi_p(-\xi x)d_p\xi. \tag{10.1}$$

Thus the operator D is hyper-singular pseudo-differential operator (PDO) with the symbol $|\xi|_p$.

Let $\alpha = 1$ be. Consider a locally-integrable in $\mathbb{Q}_p$ function

$$f_1(x) = -\frac{1-p^{-1}}{\ln p}\ln|x|_p. \tag{10.2}$$

It possesses the following properties:

$$\int f_\alpha(x)\varphi(x)d_px \to \int f_1(x)\varphi(x)d_px, \quad \alpha \to 1, \tag{10.3}$$

if $\varphi \in \mathscr{S}$ satisfies the condition

$$\int \varphi(x)d_px = 0; \tag{10.4}$$

$$\tilde{f}_1(\xi) = \operatorname{reg}|\xi|_p^{-1} + \frac{1}{p}\delta(\xi) \tag{10.5}$$

where a generalized function $\operatorname{reg}|\xi|_p^{-1}$ is defined in §6;

$$f_1 * f_\alpha = f_{1-\alpha}, \quad \alpha \geqslant 1. \tag{10.6}$$

The operator of integration of order 1 corresponding to the value of $\alpha = -1$ is equal

$$D^{-1}f = f_1 * f, \quad f \in \mathscr{S}' \tag{10.7}$$

if the convolution $f_1 * f$ exists. Then

$$D^{-\alpha}f \to D^{-1}f, \alpha \to 1 \text{ in } \mathscr{S}' \tag{10.8}$$

if $f \in \mathscr{E}'$ and

$$G\!\!\int f(x)d_px = 0. \tag{10.9}$$

Summarizing we get the following properties of the operator $D^\alpha, \alpha \in \mathbb{R}$:

$$D^\alpha D^\beta f = D^{\alpha+\beta}f = D^\beta D^\alpha f, \quad f \in \mathscr{S}' \tag{10.10}$$

if $(\alpha, \beta, \alpha+\beta) \neq (-1, -1, -1)$ or $\alpha \leqslant 0, \beta = -1$ or $\alpha = -1, \beta \leqslant 0$; if f satisfies the codition (10.9) then the equalities (10.10) are valid for all real α and β, and $D^\alpha f$ continuously depends on α in $\mathscr{S}'$.

Example.

$$D^\alpha \chi_p(ax) = |a|_p^\alpha \chi_p(ax), \quad \alpha \in \mathbb{R}, \quad a \in \mathbb{Q}_p^\times. \tag{10.11}$$

The equation

$$D^\alpha \psi = g, \quad g \in \mathscr{E}' \tag{10.12}$$

is solvable for all $\alpha \in \mathbb{R}$ and also for $\alpha > 0$ its general solution is expressed by the formula

$$\psi = D^{-\alpha} g + C \tag{10.13}$$

where C is arbitrary constant; for $\alpha \leqslant 0$ its solution is unique and it expressed by the formula (10.13) for $C = 0$.

The fundamental solution $\mathscr{E}(x)$ of the operator D^α,

$$D^\alpha \mathscr{E}(x) = \delta(x), \quad \mathscr{E} \in \mathscr{S}' \tag{10.14}$$

has been calculated in [2a)]. It is equal to

$$\mathscr{E}(x) = \begin{cases} \Gamma_p^{-1}(\alpha)|x|_p^{\alpha-1}, & \alpha \neq 1, \\ -\frac{1-p^{-1}}{\ln p} \ln |x|_p, & \alpha = 1. \end{cases} \tag{10.15}$$

Note that a fundamental solution does not exist in $\mathscr{S}'$ for any PDO. For example, for the operator $D_t^\alpha - D_x^\alpha$ it is the case. Indeed, if a solution $\mathscr{E}$ of the equation

$$(D_t^\alpha - D_x^\alpha)\mathscr{E}(t, x) = \delta(t, x)$$

would exist in $\mathscr{S}'$ so we would have the contradictory equation

$$(|\eta|_p^\alpha - |\xi|_p^\alpha)F[\mathscr{E}](\eta, \xi) = 1, \quad (\eta, \xi) \in \mathbb{Q}_p^2$$

in which left-hand side vanishes in the open set $|\eta|_p = |\xi|_p$ of space $\mathbb{Q}_p^2$.

The operator D^α for $\alpha > 0$ in a clopen set G is defined on those $\psi \in \mathscr{L}^2(G)$ (see §4) for which $|\xi|_p^\alpha \tilde{\psi} \in \mathscr{L}^2$. This set of functions is called *domain of definition* of the operator D^α in the clopen set G and it is denoted $\mathscr{D}(D^\alpha, G)$; $\quad \mathscr{D}(D^\alpha, \mathbb{Q}_p) = \mathscr{D}(D^\alpha)$. The following equality is valid

$$(D^\alpha \psi, \varphi) = \int |\xi|_p^\alpha \tilde{\psi}(\xi) \bar{\tilde{\varphi}}(\xi) d_p\xi, \quad \psi, \varphi \in \mathscr{D}(D^\alpha, G). \tag{10.16}$$

The operator D^α in G is self-adjoint positive-definite, and also owing to (10.16) for all $\psi \in \mathscr{D}(D^\alpha, G)$ we have

$$(D^\alpha \psi, \psi) = (D^{\alpha/2}\psi, D^{\alpha/2}\psi) = \int |\xi|_p^\alpha |\psi(\xi)|^2 d_p\xi \geqslant 0, \tag{10.17}$$

so its spectrum is situated on semi-axis $\lambda \geqslant 0$.

For the operator $D^\alpha, \alpha > 0$ we consider the eigen-value problem

$$D^\alpha \psi = \lambda \psi, \quad \psi \in \mathscr{D}(D^\alpha, G). \tag{10.18}$$

Theorem [1],[1b)]. *The spctrum of the operator D^α in $\mathbb{Q}_p$ consists of countable number of eigen-values $\lambda_N = p^{\alpha N}, N \in Z$ every of which is infinite multiplicity, and the point 0. There exists an ortho-normalized bases of eigen-functions in $\mathscr{L}^2(\mathbb{Q}_p)$ of the operator D^α, and it have the following form: for $p \neq 2$*

$$\psi^\ell_{N,j,\epsilon}(x) = p^{\frac{N+1-\ell}{2}} \delta(|x|_p - p^{\ell - N}) \delta(x_0 - j) \chi_p(\epsilon_\ell p^{\ell - 2N} x^2), \tag{10.19}$$

$$\ell = 2, 3, \dots, j = 1, 2, \dots, p-1, \epsilon_\ell = \varepsilon_0 + \varepsilon_1 p + \ldots + \varepsilon_{\ell-2} p^{\ell-2},$$

$$\varepsilon_s = 0, 1, \dots, p-1, \varepsilon_0 \neq 0, s = 0, 1, \dots, \ell - 2, \varepsilon_0 \neq 0, s = 0, 1, \dots, \ell - 2;$$

$$\psi^1_{N,j,0}(x) = p^{\frac{N-1}{2}} \Omega(p^{N-1}|x|_p) \chi_p(j p^{-N} x), \quad \ell = 1,$$

$$j = 1, 2, \dots, p-1, \epsilon_\ell = 0; \tag{10.19$'$}$$

for $p = 2$

$$\psi^\ell_{N,j,\epsilon_\ell}(x) = 2^{\frac{N-\ell}{2}} \delta(|x|_2 - 2^{\ell+1-N}) \chi_2(\epsilon_\ell 2^{\ell-2N} x^2 + 2^{\ell - N + j} x), \tag{10.20}$$

$$\ell = 2, 3, \dots, j = 0, 1, \epsilon_\ell = 1 + \varepsilon_1 2 + \ldots + \varepsilon_{\ell-2} 2^{\ell-2}, \varepsilon_s = 0, 1, s = 1, 2, \dots, \ell - 2;$$

$$\psi^1_{N,j,0}(x) = 2^{\frac{N-1}{2}} [\Omega(2^N |x - j 2^{N-2}|_2) - \delta(|x - j2^{N-2}|_2 - 2^{1-N})],$$

$$\ell = 1, j = 0, 1, \epsilon_\ell = 0. \tag{10.20$'$}$$

Theorem [2b)],[2c)]. *If G is a clopen compact then eigen-values $\lambda_k, k = 0, 1, \ldots$ of the operator $D^\alpha, \alpha > 0$ in G are of finite multplicity and eigen-functions $\psi_k(x)$ form an ortho-normalized bases in $\mathscr{L}^2(G)$.*

Example. Eigen-values and ortho-normalized bases of eigen-functions of the operator D^α in $B_\gamma, \gamma \in Z$ [2b)]. For $p \neq 2$:

$$\lambda_0 = \frac{p-1}{p^{\alpha+1} - 1} p^{\alpha(1-\gamma)}, \quad \psi_0(x) = p^{-\gamma/2}, \text{ multipl. } 1;$$

$$\lambda_k = p^{\alpha(k-\gamma)}, \quad \psi_k(x) = \psi^\ell_{k-\gamma, j, \epsilon_\ell}(x), \quad \ell = 1, 2, \dots, k, j = 1, 2, \dots, p-1, \epsilon_\ell,$$

$$\text{multipl. } (p-1)p^{k-1}, \quad k = 1, 2, \dots .$$

For $p = 2$:

$$\lambda_0 = \frac{2^{\alpha(1-\gamma)}}{2^{\alpha+1}-1}, \quad \psi_0(x) = 2^{-\gamma/2}, \text{ multipl. } 1;$$

$$\lambda_1 = 2^{\alpha(1-\gamma)}, \quad \psi_1(x) = \psi^1_{1-\gamma,0,0}(x), \text{ multipl. } 1;$$

$$\lambda_k = 2^{\alpha(k-\gamma)}, \quad \psi_k(x) = \psi^\ell_{k-\gamma,j,\epsilon_\ell}(x), \quad \ell = 1,2,\dots,k-1, j = 0,1,$$

$$\text{multipl. } 2^{k-1}, \quad k = 2,3,\dots.$$

Example. Eigen-values and normalized bases of eigen-functions of the operator D^α in $S_\gamma, \gamma \in Z$ [2b)]. For $p \neq 2$:

$$\lambda_0 = \frac{p^\alpha + p - 2}{p^{\alpha+1}-1} p^{\alpha(1-\gamma)}, \quad \psi_0(x) = p^{\frac{1-\gamma}{2}}(p-1)^{1/2}, \text{ multipl. } 1;$$

$$\lambda_1 = p^{\alpha(1-\gamma)}, \quad \psi_1(x) = 2^{-1/2}[\psi^1_{1-\gamma,j,0}(x) - \psi^1_{1-\gamma,j+1,0}(x)], \text{ multipl. } p-2;$$

$$\lambda_k = p^{\alpha(k-\gamma)}, \quad \psi_k(x) = \psi^k_{k-\gamma,j,\epsilon_k}(x), \quad j = 1,2,\dots,p-1, \quad \epsilon_k,$$

$$\text{multipl. } (p-1)^2 p^{k-2}, \quad k = 2,3,\dots.$$

For $p = 2$:

$$\lambda_0 = \frac{2^{\alpha(2-\gamma)}}{2^{\alpha+1}-1}, \quad \psi_0(x) = 2^{\frac{1-\gamma}{2}}, \text{ multipl. } 1;$$

$$\lambda_1 = 2^{\alpha(2-\gamma)}, \quad \psi_1(x) = \psi^1_{1-\gamma,1,0}(x), \text{ multipl. } 1;$$

$$\lambda_k = 2^{\alpha(k+1-\gamma)}, \quad \psi_k(x) = \psi^k_{k+1-\gamma,j,\epsilon_k}(x), \quad j = 0,1, \quad \epsilon_k,$$

$$\text{multipl. } 2^{k-1}, \quad k = 2,3,\dots.$$

It should be pointed out that multiplicative characters of rank k of the group $Z_p^\times$ are eigen-functions of the operator D^α in S_0 correspondig to the eigen-value λ_k [11a)]. On the other hand, a number of linearly idependent multiplicative characters of rank k of the group $Z_p^\times$ was calculated (see [16]) and it coincides to the multiplicity n_k of the eigen-value λ_k f the operator $D^\alpha, \alpha > 0$ in S_0 [2b)]. From here it follows such result:

There is exist an ortho-nomalized bases of eigen-functions of the operator $D^\alpha, \alpha > 0$ in S_0 consisting of all multiplicative characters of the group $Z_p^\times$.

On the other hand, any multiplicative character of the group $Z_p^\times$ of rank k is expanded on eigen-functions $\psi_{a_k+j}(x), j = 1,2,\dots,n_k$ (by a suitable

choose of a_k [2b)], that is it is expanded on additive characters of the field $\mathbb{Q}_p$.

Indicate concrete values for λ_k and n_k. Assuming $\gamma = 0$ we get [2b)]: for $p \neq 2$

$$\lambda_0 = \frac{p^\alpha + p - 2}{p^{\alpha+1} - 1} p^\alpha, \quad n_0 = 1;$$

$$\lambda_1 = p^\alpha, \quad n_1 = p - 2; \quad \lambda_k = p^{\alpha k}, \quad n_k = (p-1)^2 p^{k-2}, \quad k = 2, 3, \dots ;$$

for $p = 2$

$$\lambda_0 = \frac{2^{2\alpha}}{2^{\alpha+1} - 1}, \quad n_0 = 1; \quad \lambda_k = 2^{\alpha(k+1)}, \quad n_k = 2^{k-1}, \quad k = 1, 2, \dots .$$

Part II
Tables of integrals

§11. Primary integrals, one variable

$$\int_{B_0} d_p x = 1. \tag{11.1}$$

$$\int_{B_\gamma} d_p x = p^\gamma. \tag{11.2}$$

$$\int_{S_\gamma} d_p x = (1 - 1/p)p^\gamma. \tag{11.3}$$

$$\int f(x) d_p x = \sum_{\gamma=-\infty}^{\infty} \int_{S_\gamma} f(x) d_p x. \tag{11.4}$$

$$\int_{B_\gamma} f(|x|_p) d_p x = (1 - 1/p) \sum_{k=-\infty}^{\gamma} p^k f(p^k). \tag{11.5}$$

$$\int f(|x|_p) d_p x = (1 - 1/p) \sum_{k=-\infty}^{\infty} p^k f(p^k). \tag{11.6}$$

$$\int_D f(x) d_p x = |a|_p \int_{\frac{D-b}{a}} f(ay + b) d_p y, \quad a \neq 0. \tag{11.7}$$

$$\int_{S_\gamma} f(x) d_p x = p^{2\gamma} \int_{S_{-\gamma}} f(1/y) d_p y. \tag{11.8}$$

$$\int_{B_\gamma} f(x) d_p x = \int_{\mathbb{Q}_p \setminus B_{1-\gamma}} f(1/y) |y|_p^{-2} d_p y. \tag{11.9}$$

$$\int f(x) d_p x = \int f(1/y) |y|_p^{-2} d_p y. \tag{11.10}$$

$$\int f(|x|_p) d_p x = \int f(1/|y|_p) |y|_p^{-2} d_p y. \tag{11.11}$$

$$\int_{G_p} f(x) d_p x = \int_{G_p} f(\sin y) d_p y. \tag{11.12}$$

$$\int_{G_p} f(x)d_px = \int_{G_p} f(\arcsin y)d_py. \tag{11.13}$$

$$\int_{G_p} f(x)d_px = \int_{G_p} f(\operatorname{tg} y)d_py. \tag{11.14}$$

$$\int_{G_p} f(x)d_px = \int_{G_p} f(\operatorname{arctg} y)d_py. \tag{11.15}$$

$$\int_{G_p} f(x)d_px = \int_{J_p} f(\ln y)d_py. \tag{11.16}$$

$$\int_{J_p} f(x)d_px = \int_{G_p} f(\exp y)d_py. \tag{11.17}$$

$$\int_{B_\gamma} |x|_p^{\alpha-1}d_px = \frac{1-p^{-1}}{1-p^{-\alpha}}p^{\alpha\gamma}, \quad \operatorname{Re}\alpha > 0. \tag{11.18}$$

$$\int_{S_0} |x-1|_p^{\alpha-1}d_px = \frac{p-2+p^{-\alpha}}{p(1-p^{-\alpha})}, \quad \operatorname{Re}\alpha > 0 \quad [2a)]. \tag{11.19}$$

.

$$\int_{S_\gamma} |x-a|_p^{\alpha-1}d_px = \frac{p-2+p^{-\alpha}}{p(1-p^{-\alpha})}|a|_p^{\alpha}, \quad |a|_p = p^\gamma, \operatorname{Re}\alpha > 0. \tag{11.20}$$

$$\int_{B_\gamma} \ln|x|_p d_px = \Big(\gamma - \frac{1}{p-1}\Big)p^\gamma \ln p. \tag{11.21}$$

$$\int_{S_0} \ln|x-1|_p d_px = -\frac{\ln p}{p-1} \quad [2a)]. \tag{11.22}$$

$$\int_{S_\gamma} \ln|x-a|_p d_px = \Big[(1-1/p)\ln|a|_p - \frac{\ln p}{p-1}\Big]|a|_p, \quad |a|_p = p^\gamma. \tag{11.23}$$

$$\int_{S_\gamma} \ln|x|_p d_px = \gamma(1-1/p)p^\gamma \ln p. \tag{11.24}$$

$$\int |x|_p^{\alpha-1}|1-x|_p^{\beta-1}d_px = B_p(\alpha,\beta),$$

$$\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}(\alpha+\beta) < 1 \quad [4]. \tag{11.25}$$

$$\int |x|_p^{\alpha-1}|y-x|_p^{\beta-1}d_px = B_p(\alpha,\beta)|y|_p^{\alpha+\beta-1},$$

$$\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}(\alpha+\beta) < 1. \tag{11.26}$$

$$\int_{B_\gamma} |x^2+a^2|_p^{(\alpha-1)/2} d_p x = p^\gamma |a|_p^{\alpha-1}, \quad p^\gamma < |a|_p. \tag{11.27}$$

$$= \frac{1-p^{\alpha-1}}{1-p^\alpha}|a|_p^\alpha + \frac{1-p^{-1}}{1-p^{-\alpha}} p^{\alpha\gamma},$$

$$p^\gamma \geqslant |a|_p \neq 0, \operatorname{Re}\alpha > 0, p \equiv 3 (\operatorname{mod} 4) \quad [3a)]. \tag{11.28}$$

$$= \Big[1 - 2/p + \Big(1-1/p\Big)\Big(\frac{2}{p^{(\alpha+1)/2}-1} - \frac{1}{1-p^{-\alpha}}\Big)\Big]|a|_p^\alpha - \frac{1-p^{-1}}{1-p^{-\alpha}}p^{\alpha\gamma},$$

$$p^\gamma \geqslant |a|_p \neq 0, \operatorname{Re}\alpha > 0, p \equiv 1 (\operatorname{mod} 4) \quad [3a)]. \tag{11.29}$$

$$\int |x^2+a^2|_p^{(\alpha-1)/2} d_p x, \quad a \neq 0$$

$$= \frac{1-p^{\alpha-1}}{1-p^\alpha}|a|_p^\alpha, \quad \operatorname{Re}\alpha < 0, p \equiv 3 (\operatorname{mod} 4). \tag{11.30}$$

$$= \Big[1 - 2/p + \Big(1-1/p\Big)\Big(\frac{2}{p^{(\alpha+1)/2}-1} - \frac{1}{1-p^{-\alpha}}\Big)\Big]|a|_p^\alpha,$$

$$\operatorname{Re}\alpha < 0, p \equiv 1 (\operatorname{mod} 4). \tag{11.31}$$

$$\int_{S_0} |1+x^2|_p^{\alpha-1} d_p x = 1 - 3/p - 2\frac{1-p^{-1}}{1-p^\alpha}, \quad \operatorname{Re}\alpha > 0, p \equiv 1 (\operatorname{mod} 4). \tag{11.32}$$

$$\int_{S_{\gamma,k_0}} d_p x = p^{\gamma-1}, \quad k_0 = 1, 2, \dots, p-1 \quad [2a)]. \tag{11.33}$$

$$\int_{S_\gamma^{k_0}} d_p x = (1-2/p)p^\gamma, \quad k_0 = 1, 2, \dots, p-1 \quad [2a)]. \tag{11.34}$$

$$\int_{S_{\gamma,k_n}} d_p x = (1-1/p)p^{\gamma-1}, \quad k_n = 0, 1, \dots, p-1, n \in Z_+ \quad [2a)]. \tag{11.35}$$

$$\int_{S_\gamma^{k_n}} d_p x = (1-1/p)^2 p^\gamma, \quad k_n = 0, 1, \dots, p-1, n \in Z_+ \quad [2a)]. \tag{11.36}$$

$$\int_{S_{\gamma,k_0 k_1 \dots k_n}} d_p x = p^{\gamma-n-1},$$

$$k_j = 0, 1, \dots, p-1, k_0 \neq 0, n \in Z_+ \quad [2a)]. \tag{11.37}$$

$$\int_{S_\gamma^{k_0k_1\dots k_n}} d_px = (1-p^{-1}-p^{-n-1})p^\gamma,$$

$$k_j = 0.,1,\dots,p-1, k_0 \neq 0, n \in Z_+ \quad [2a)]. \tag{11.38}$$

$$\int_{\cap_{1\leqslant i\leqslant k}[|x-x_i|_p=1]} d_px = 1-k/p,$$

$$1 \leqslant k \leqslant p, |x_j - x_j|_p = 1, i,j = 1,2,\dots,k, i \neq j \quad [9a)]. \tag{11.39}$$

Let π be a multiplicative character of the field $\mathbb{Q}_p$ of rank $k \geqslant 1$.

$$\int_{S_\gamma} \pi(x)d_px = 0 \quad [4]. \tag{11.40}$$

Denote: $V_0 = S_0, V_j = [x \in S_0 : |1-x|_p \leqslant p^{-j}], j \in Z_+$.

$$\int_{V_j\setminus V_{j+1}} \pi(x)d_px = 0, \quad 0 \leqslant j < k-1. \tag{11.41}$$

$$= -p^{-k}, \quad j = k-1. \tag{11.42}$$

$$= (1-1/p)p^{-j}, \quad j \geqslant k \quad [11a)]. \tag{11.43}$$

$$\int_{S_0} |1-x|_p^{\alpha-1}\pi(x)d_px = \Gamma_p(\alpha)p^{-k\alpha}, \quad \operatorname{Re}\alpha > 0 \quad [11a)]. \tag{11.44}$$

$$\int_{S_\gamma} \operatorname{sgn}_{p,\epsilon} x d_px = (1-1/p)(-p)^\gamma, \quad \epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1, p \neq 2 \quad [4]. \tag{11.45}$$

$$\int_{B_\gamma} \operatorname{sgn}_{p,\epsilon} x d_px = \frac{p-1}{p+1}(-p)^\gamma, \quad \epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1, p \neq 2 \quad [4]. \tag{11.46}$$

$$\int_{B_0} \operatorname{sgn}_{p,\epsilon} x d_px = \frac{p-1}{p+1}, \quad \epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1, p \neq 2 \quad [4]. \tag{11.47}$$

$$= 1/3, \quad \epsilon \equiv 5 (\operatorname{mod} 8), p = 2, \tag{11.48}$$

$$= 0, \quad |\epsilon|_p = 1/p, p \neq 2 \text{ or } \epsilon \not\equiv 1,5 (\operatorname{mod} 8), p = 2 \quad [4]. \tag{11.49}$$

$$\int_{B_0} \theta_\epsilon^+(x)d_px = \frac{p}{p+1}, \quad \epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1, p \neq 2. \tag{11.50}$$

$$= 2/3, \quad \epsilon \equiv 5 (\operatorname{mod} 8), p = 2. \tag{11.51}$$

$$= 1/2, \quad |\epsilon|_p = 1/p, p \neq 2 \text{ or } \epsilon \not\equiv 1,5 (\operatorname{mod} 8), p = 2 \quad [4]. \tag{11.52}$$

$$\int_{B_0} \theta_\epsilon^-(x) d_p x = \frac{1}{p+1}, \quad \epsilon \notin \mathbb{Q}_p^{\times 2}, |\epsilon|_p = 1, p \neq 2. \tag{11.53}$$

$$= 1/3, \quad \epsilon \equiv 5 (\mathrm{mod}\, 8), p = 2. \tag{11.54}$$

$$= 1/2, \quad |\epsilon|_p = 1/p, p \neq 2 \text{ or } \epsilon \not\equiv 1, 5 (\mathrm{mod}\, 8), p = 2 \quad [4]. \tag{11.55}$$

$$\int_{(B_0)^2} d_p x = \frac{p}{2(p+1)}, \quad p \neq 2. \tag{11.56}$$

$$= 1/6, \quad p = 2 \tag{11.57}$$

where $(B_0)^2$ is the set of squares of integers p-adic numbers Z_p.

$$\int_{\gamma(x)=2k\leqslant 0} d_p x = \frac{p}{p+1}. \tag{11.58}$$

$$\int_{\gamma(x)=2k\leqslant 0} f(|x|_p) d_p x = (1 - 1/p) \sum_{\gamma=0}^{\infty} p^{-2\gamma} f(p^{-2\gamma}). \tag{11.59}$$

$$\int_{\gamma(x)-1=2k\leqslant 0} d_p x = \frac{1}{p+1}. \tag{11.60}$$

$$\int_{\gamma(x)-1=2k\leqslant 0} f(|x|_p) d_p x = (1 - 1/p) \sum_{\gamma=0}^{\infty} p^{-2\gamma-1} f(p^{-2\gamma-1}). \tag{11.61}$$

$$\int_{B_0} \lambda_p(x) |x|_p^{-1/2} d_p x = 1, \quad p \neq 2 \quad [1b)]. \tag{11.62}$$

$$= 2^{-3/2}, \quad p = 2 \quad [1b)]. \tag{11.63}$$

Let a function f has the property

$$\int_{B_0} f(x+k) d_p x = f(k), \quad k \in I_p$$

where I_p is the set of indexes,

$$I_p = [k \in \mathbb{Q}_p : k = p^{-\gamma}(k_0 + k_1 + \ldots + k_{\gamma-1} p^{\gamma-1}),$$

$$k_j = 0, 1, \ldots, p-1, k_0 \neq 0, j = 0, 1, \ldots, \gamma - 1, \gamma \in Z_+].$$

$$\int_{\mathbb{Q}_p\setminus B_0} f(x)d_px = \sum_{k\in I_p} f(k) \quad [14]. \tag{11.64}$$

$$\int_{B_{-1}\setminus B_{-2n}} \lambda_p^2(x)|x|_p^{-1}d_px, \quad n\in Z_+$$
$$= 1-1/p, \quad p\equiv 3(\mathrm{mod}\,4) \quad [1b)] \tag{11.65}$$
$$= (1-1/p)(2n-1), \quad p\equiv 1(\mathrm{mod}\,4) \quad [1b)] \tag{11.66}$$

$$\int_{B_{-2}\setminus B_{-2n}} \lambda_2^2(x)|x|_2^{-1}d_2x = 0, \quad p=2, n\geqslant 2 \quad [1b)]. \tag{11.67}$$

Denote $|(x,m)|_p = \max(|x|_p, |m|_p)$.

$$\int |(y,m)|_p^{\alpha-1}|(x-y,m)|_p^{\beta-1}d_py = B_p(\alpha,\beta)|(x,m)|_p^{\alpha+\beta-1}$$
$$-\Gamma_p(\alpha)|pm|_p^{\alpha}|(x,m)|_p^{\beta-1} - \Gamma_p(\beta)|pm|_p^{\beta}|(x,m)|_p^{\alpha-1},$$
$$m\neq 0, \mathrm{Re}(\alpha+\beta)<1 \quad [9a)]. \tag{11.68}$$

Denote:

$$\mathscr{K}_t(x,y) = \lambda_p(t)\sqrt{|2/t|_p}\chi_p\Big(\frac{2xy}{\sin t} - \frac{x^2+y^2}{\operatorname{tg} t}\Big), \quad t\in G_p, x,y\in\mathbb{Q}_p,$$

$$\mathscr{K}_t(x) = \lambda_p(t)\sqrt{|2/t|_p}\chi_p\big(-x^2/t\big), \quad t\in\mathbb{Q}_p^{\times}, x\in\mathbb{Q}_p.$$

$$\int \mathscr{K}_t(x,y')\mathscr{K}_\tau(y',x)d_py' = \mathscr{K}_{t+\tau}(x,y),$$
$$t,\tau\in G_p, x,y\in\mathbb{Q}_p \quad [7]. \tag{11.69}$$

$$\int_{B_0} \mathscr{K}_t(x,y)d_py = \Omega(|x|_p), \quad t\in G_p, x\in\mathbb{Q}_p \quad [7]. \tag{11.70}$$

$$\mathscr{K}_t(x,y)\to\delta(x-y), t\to 0 \;\; \mathscr{S}'(\mathbb{Q}_p^2) \quad [7]. \tag{11.71}$$

$$\int \mathscr{K}_t(x-y)\mathscr{K}_\tau(y)d_py = \mathscr{K}_{t+\tau}(x), \quad t,\tau\in\mathbb{Q}_p^{\times}, x\in\mathbb{Q}_p \quad [7]. \tag{11.72}$$

$$\mathscr{K}_t(x)\to\delta(x), t\to 0 \;\; \mathscr{S}' \quad [7]. \tag{11.73}$$

$$\int_{|x|_p\neq 1} f(|x|_p)|1-x|_p^{-1} d_p x = (1-p^{-1}) \sum_{\gamma\neq 0} f(p^\gamma) \min(1,p^\gamma). \qquad (11.74)$$

§12. The Fourier integrals

The Fourier integral is called an integral of the form

$$\int f(x)\chi_p(\xi x) d_p x, \quad \xi \in \mathbb{Q}_p^\times.$$

$$\int_{B_\gamma} \chi_p(\xi x) d_p x = p^\gamma \Omega(p^\gamma |\xi|_p) \quad [2a)]. \qquad (12.1)$$

$$\int_{S_\gamma} \chi_p(\xi x) d_p x = (1-1/p)p^\gamma \Omega(p^\gamma|\xi|_p) - p^{\gamma-1}\delta(|\xi|_p - p^{1-\gamma}) \quad [2a)]. \qquad (12.2)$$

$$\int \chi_p(\xi x) d_p x = 0, \quad \xi \neq 0 \quad [2a)]. \qquad (12.3)$$

$$\int_{B_\gamma} f(|x|_p)\chi_p(\xi x) d_p x$$

$$= (1-1/p) \sum_{k=-\gamma}^{\infty} p^{-k} f(p^{-k}), \quad |\xi|_p \leqslant p^{-\gamma}. \qquad (12.4)$$

$$= (1-1/p)|\xi|_p^{-1} \sum_{k=0}^{\infty} p^{-k} f(p^{-k}|\xi|_p^{-1}) - |\xi|_p^{-1} f(p|\xi|_p^{-1}),$$

$$|\xi|_p > p^{-\gamma} \quad [2a)]. \qquad (12.5)$$

$$= \int f(|x|_p)\chi_p(\xi x) d_p x, \quad |\xi|_p > p^{-\gamma}. \qquad (12.6)$$

$$\int f(|x|_p)\chi_p(\xi x) d_p x, \quad \xi \neq 0$$

$$= (1-1/p)|\xi|_p^{-1} \sum_{k=0}^{\infty} p^{-k} f(p^{-k}|\xi|_p^{-1}) - |\xi|_p^{-1} f(p|\xi|_p^{-1}) \quad [2a)]. \qquad (12.7)$$

$$\int |x|_p^{\alpha-1}\chi_p(x) d_p x = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}} = \Gamma_p(\alpha), \quad \mathrm{Re}\,\alpha > 0 \quad [4]. \qquad (12.8)$$

$$\int |x|_p^{\alpha-1}\chi_p(\xi x)d_px = \Gamma_p(\alpha)|\xi|_p^{-\alpha}, \quad \xi \neq 0, \mathrm{Re}\,\alpha > 0 \quad [4]. \tag{12.9}$$

$$\int \ln|x|_p\chi_p(x)d_px = -(1-1/p)^{-1}\ln p \quad [2a)]. \tag{12.10}$$

$$\int \ln|x|_p\chi_p(\xi x)d_px = -(1-1/p)^{-1}\ln p|\xi|_p^{-1}, \quad \xi \neq 0 \quad [2a)]. \tag{12.11}$$

$$\int \frac{\chi_p(\xi x)}{|x|_p^2+m^2}d_px, \quad m \neq 0$$

$$= (1-1/p)\sum_{k=-\infty}^{\infty}\frac{p^k}{p^{2k}+m^2}, \quad \xi = 0. \tag{12.12}$$

$$= (1-1/p)\frac{|\xi|_p}{p^2+m^2|\xi|_p^2}\sum_{k=0}^{\infty}p^{-k}\frac{p^2-p^{-2k}}{p^{-2k}+m^2|\xi|_p^2}, \quad \xi \neq 0 \quad [2a)]. \tag{12.13}$$

$$\sim \frac{p^4+p^3}{p^2+p+1}m^{-4}|\xi|_p^{-3}+O(|\xi|_p^{-5}), \quad |\xi|_p \to \infty \quad [2a)]. \tag{12.14}$$

$$\mu_t^\alpha(x) = \int \exp(-t|\xi|_p^\alpha)\chi_p(\xi x)d_p\xi, \quad t>0, \alpha>0$$

$$= (1-1/p)|x|_p^{-1}\sum_{\gamma=0}^{\infty}p^{-\gamma}\exp(-t|px|_p^{-\alpha})$$

$$\times\Big(\exp[t|px|_p^{-\alpha}(1-p^{-\alpha\gamma-\alpha})]-1\Big) > 0. \tag{12.15}$$

$$= \sum_{n=1}^{\infty}\frac{(-t)^n}{n!}\Gamma_p(\alpha n+1)|\xi|_p^{-\alpha n-1} \quad [1a)]. \tag{12.16}$$

$$\int \mu_t^\alpha(x)d_px = 1, \quad t>0. \tag{12.17}$$

$$\mu_t^\alpha(x) \to \delta(x), \quad t \to 0 \text{ in } \mathscr{S}' \quad [1a)]. \tag{12.18}$$

$$\mu_t^\alpha * \mu_\tau^\alpha = \mu_{t+\tau}^\alpha, \quad t,\tau>0 \quad [1a)]. \tag{12.19}$$

$$\int_0^\infty \mu_t^\alpha(x)dt = \Gamma_p^{-1}(\alpha)|x|_p^{\alpha-1} = f_\alpha(x), \quad \alpha \neq \alpha_k, k \in Z. \tag{12.20}$$

$$\frac{\partial}{\partial t}\mu_p^\alpha(x)|_{t=0} = \Gamma_p(\alpha+1)|x|_p^{-\alpha-1}, \quad \alpha>0. \tag{12.21}$$

$$|x|_p^\alpha = -\Gamma_p^{-1}(-\alpha)\int [1-\operatorname{Re}\chi_p(x\xi)]|\xi|_p^{-\alpha-1}d_p\xi \tag{12.22}$$

and also

$$-\Gamma_p^{-1}(-\alpha)|\xi|_p^{-\alpha-1}d_p\xi > 0, \alpha > 0.$$

$$\int_{B_{-1}} \chi_p(a^2 \operatorname{tg}\xi - x\xi)d_p\xi, \quad p \neq 2 \quad [1b)]$$

$$= 1/2\Omega(|px|_p), \quad |a|_p \leqslant 1. \tag{12.23}$$

$$= 1/2\delta(|x|_p - p^2)\delta(x_0 - (a^2)_0), \quad |a|_p = p. \tag{12.24}$$

$$= 1/2\delta(|x|_p - |a|_p^2)\delta(x_0 - (a^2)_0)\delta(x_1 - (a^2)_1)\varphi_a(x),$$

$$|a|_p \geqslant p^2 \tag{12.25}$$

where $\varphi_a(x)$ is a continuous function.

$$\int |x|_p^{\alpha-1}|x-a|_p^{\beta-1}\chi_p(x)d_px, \quad \operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}(\alpha+\beta) < 1$$

$$= B_p(\alpha,\beta)|a|_p^{\alpha+\beta-1} + \Gamma_p(\alpha+\beta-1), \quad |a|_p \leqslant 1. \tag{12.26}$$

$$= \Gamma_p(\alpha)|a|_p^{\beta-1} + \Gamma_p(\beta)|a|_p^{\alpha-1}\chi_p(a), \quad |a|_p \geqslant p. \tag{12.27}$$

$$\int_{S_\gamma} |x-a|_p^{\alpha-1}\chi_p(x-a)d_px = \Gamma_p(\alpha),$$

$$|a|_p = p^\gamma, \gamma \geqslant 2, \operatorname{Re}\alpha > 0. \tag{12.28}$$

Let $n \in Z_+$ be not divisible by p and P be a polynom of degree n,

$$P(x) = \alpha_1 x + \alpha_2 x^2 + \ldots + \alpha_n x^n, |\alpha_k|_p \leqslant 1, k = 1, 2, \ldots, n-1, |\alpha_n|_p = 1.$$

$$\int_{S_\gamma} \chi_p[P(x)]d_px = (1-1/p)p^\gamma, \quad \gamma \leqslant 0, n \in Z_+ \quad [2a)]. \tag{12.29}$$

$$= 0, \quad \gamma = 2, 3, \ldots, n \in Z_+ \;\; \gamma = 1, n = 2, 3, \ldots \quad [2a)]. \tag{12.30}$$

$$= -1, \quad \gamma = 1, n = 1 \quad [2a)]. \tag{12.31}$$

$$\int_{B_\gamma} \chi_p[P(x)]d_px = p^\gamma, \quad \gamma \leqslant 0, n \in Z_+. \tag{12.32}$$

$$= 1, \quad \gamma = 2, 3, \ldots, n \in Z_+ \;\; \gamma = 1, n = 2, 3, \ldots. \tag{12.33}$$

$$= 0, \quad \gamma \in Z_+, n = 1. \tag{12.34}$$

$$\int \chi_p[P(x)]d_px = 1, \quad n = 2, 3, \dots . \tag{12.35}$$

$$= 0, \quad n = 1. \tag{12.36}$$

Let (complex) numbers $\eta_1, \eta_2, \dots, \eta_{p-1}$ be such that

$$\sum_{k=1}^{p-1} \eta_k = 0, \quad p \neq 2,$$

and numbers $\eta'_1, \eta'_2, \dots, \eta'_{p-1}$ are mutual to $\{\eta_k\}, k = 1, 2, \dots, p-1$,

$$\eta'_j = \sum_{k=1}^{p-1} \eta_k \exp(2\pi ikj/p), \quad \sum_{j=1}^{p-1} \eta'_j = 0.$$

$$\int_{S_\gamma} \eta_{x_0} \chi_p(\xi x) d_px = p^{\gamma-1} \eta'_{\xi_0} \delta(|\xi|_p - p^{1-\gamma}) \quad [2b)]. \tag{12.37}$$

$$\int_{B_\gamma} |x|_p^{\alpha-1} \chi_p(\xi x) d_px, \quad \mathrm{Re}\, \alpha > 0$$

$$= \frac{1 - p{-}1}{1 - p^{-\alpha}} p^{\alpha\gamma}, \quad |\xi|_p \leqslant p^{-\gamma}. \tag{12.38}$$

$$= \Gamma_p(\alpha)|\xi|_p^{-\alpha}, \quad |\xi|_p > p^{-\gamma} \quad [1a)]. \tag{12.39}$$

$$= \Gamma_p(\alpha), \quad \xi = 1, \gamma \geqslant 1 \quad [2e)]. \tag{12.40}$$

$$\int_{S_0} \delta(x_0 - k) \chi_p(\xi x) d_px$$

$$= p^{-1} \chi_p(k\xi) \Omega(|p\xi|_p), \quad k = 1, 2, \dots, p-1. \tag{12.41}$$

$$\int \delta(x_0 - k) \chi_p(\xi x) d_px$$

$$= |\xi|_p^{-1} \Big(\frac{1}{p-1} + \chi_p(k\xi_0/p) \Big), \quad \xi \neq 0, \quad k = 1, 2, \dots, p-1. \tag{12.42}$$

$$\int_{B_1} \chi_p[(k - \xi)x] d_px = p\delta(|\xi|_p - 1)\delta(\xi_0 - k), \quad k = 1, 2, \dots, p-1. \tag{12.43}$$

$$\int_{S_0} \delta(x_1 - k) \chi_p(\xi x) d_px = 1/p(1 - 1/p)\Omega(|\xi|_p) - p^{-2}\delta(|\xi|_p - p)$$

$$+p^{-2}\frac{\chi_p(\xi)-\chi_p(p\xi)}{1-\chi_p(\xi)}\chi_p(kp\xi)\delta(|\xi|_p-p^2), \quad k=0,1,\dots,p-1. \tag{12.44}$$

$$\int \delta(x_1-k)\chi_p(\xi x)d_px$$

$$=|\xi|_p^{-1}\frac{\chi_p(p^{-2}|\xi|_p\xi)-\chi_p(p^{-1}\xi_0)}{1-\chi_p(p^{-1}\xi_0)}\chi_p(kp^{-1}\xi_0),$$

$$\xi\neq 0, p=0,1,\dots,p-1. \tag{12.45}$$

$$\int_{S_0}\delta(x_2-k)\chi_p(\xi x)d_px = 1/p(1-1/p)\Omega(|\xi|_p)-p^{-2}\delta(|\xi|_p-p)$$

$$+p^{-3}\frac{\chi_p(\xi)-\chi_p(p\xi)}{1-\chi_p(\xi)}\frac{\chi_p(kp^2\xi)-\chi_p((k+1)p^2\xi)}{1-\chi_p(p\xi)}\delta(|\xi|_p-p^3),$$

$$k=0,1,\dots,p-1. \tag{12.46}$$

$$\int \delta(x_2-k)\chi_p(\xi x)d_px$$

$$=|\xi|_p^{-1}\frac{\chi_p(p^{-3}|\xi|_p\xi)-\chi_p(p^{-2}|\xi|_p\xi)}{1-\chi_p(p^{-3}|\xi|_p\xi)}\frac{\chi_p(kp^{-1}\xi_0)-\chi_p((k+1)p^{-1}\xi_0)}{1-\chi_p(p^{-2}|\xi|_p\xi)},$$

$$\xi\neq 0, k=0,1,\dots,p-1. \tag{12.47}$$

$$\int |x,m|_p^{\alpha-1}\chi_p(\xi x)d_px$$

$$=\Gamma_p(\alpha)\big(|\xi|_p^{-\alpha}-|pm|_p^{\alpha}\big)\Omega(|m\xi|_p), \quad m\neq 0, \operatorname{Re}\alpha<0 \quad [9a)]. \tag{12.48}$$

$$\int |x,1|_p^{-\alpha}\chi_p(\xi x)d_px$$

$$=\Gamma_p(1-\alpha)(|\xi|_p^{\alpha-1}-p^{\alpha-1})\Omega(|\xi|_p)\equiv J_p^{\alpha}(\xi), \quad \operatorname{Re}\alpha>0. \tag{12.49}$$

$$J_p^1(\xi)=(1-1/p)\left(1-\frac{\ln|\xi|_p}{\ln p}\right)\Omega(|\xi|_p), \quad \alpha=1. \tag{12.50}$$

$$\int J_p^{\alpha}(\xi)J_p^{\beta}(x-\xi)d_p\xi = J_p^{\alpha}*J_p^{\beta}=J_p^{\alpha+\beta}, \quad \alpha,\beta\in\mathbb{C}. \tag{12.51}$$

$$\ln|x,1|_p=\int\big(1-\operatorname{Re}\chi_p(x\xi)\big)d\sigma(\xi)$$

$$= \ln p \sum_{\gamma=0}^{\infty} p^{\gamma}\Omega(p^{\gamma}|\xi|_p), \quad d\sigma(\xi) \geqslant 0. \tag{12.52}$$

$$\int_{B_{-1}\setminus B_{-2n}} \lambda_p^2(x)|x|_p^{-1}\chi_p(\xi x)d_px, \quad n \in Z_+$$

$$= (1-1/p)(2n-1), \quad \xi = 0 \;\; \gamma(\xi) \leqslant 1, p \equiv 1(\mathrm{mod}\,4) \quad [1b]. \tag{12.53}$$

$$= (1-1/p)(2n-\gamma(\xi)) - 1/p,$$
$$2 \leqslant \gamma(\xi) \leqslant 2n, p \equiv 1(\mathrm{mod}\,4) \quad [1b]. \tag{12.54}$$

$$= 1-1/p, \quad \xi = 0 \;\; \gamma(\xi) \leqslant 1, p \equiv 3(\mathrm{mod}\,4) \quad [1b]. \tag{12.55}$$

$$= 1/2(-1)^{\gamma(\xi)}(1+1/p) - 1/2(1-1/p),$$
$$2 \leqslant \gamma(\xi) \leqslant 2n, p \equiv 3(\mathrm{mod}\,4) \quad [1b]. \tag{12.56}$$

$$\int_{B_{-1}} \lambda_p^2(x)|x|_p^{-1}\chi_p(\xi x)d_px$$

$$= 1-1/p, \quad \xi = 0 \;\; \gamma(\xi) \leqslant 1, p \equiv 3(\mathrm{mod}\,4) \quad [1b]. \tag{12.57}$$

$$= 1/2(-1)^{\gamma(\xi)}(1+1/p) - 1/2(1-1/p),$$
$$\gamma(\xi) \geqslant 2, p \equiv 3(\mathrm{mod}\,4) \quad [1b]. \tag{12.58}$$

$$\int_{B_{-2}\setminus B_{-2n}} \lambda_2^2(x)|x|_2^{-1}\chi_2(\xi x)d_2x, \quad n = 2, 3, \ldots$$

$$= 0, \quad \gamma(\xi) \leqslant 3 \quad [1b]. \tag{12.59}$$

$$= 1/4(-1)^{\xi_1+1}, \quad \gamma(\xi) \geqslant 4 \quad [1b]. \tag{12.60}$$

$$\int_{B_{-2}} \lambda_2^2(x)|x|_2^{-1}\chi_2(\xi x)d_2x = 0, \quad \xi = 0 \text{ or } \gamma(\xi) \leqslant 3 \quad [1b]. \tag{12.61}$$

$$= 1/2(-1)^{\xi_1+1}, \quad \gamma(\xi) \geqslant 4 \quad [1b]. \tag{12.62}$$

$$\int \mathrm{sgn}_{p,d} x|x|_p^{\alpha-1}\chi_p(\xi x)d_px, \quad d \notin \mathbb{Q}_p^{\times 2}$$

$$= \tilde{\Gamma}_p(\alpha)\mathrm{sgn}_{p,d}\xi|\xi|_p^{-\alpha}, \quad |d|_p = 1, \mathrm{Re}\,\alpha > 0 \quad [4]. \tag{12.63}$$

$$= \pm p^{\alpha-1/2}\sqrt{\mathrm{sgn}_{p,d}(-1)}\mathrm{sgn}_{p,d}\xi|\xi|_p^{-\alpha}, \quad |d|_p = 1/p, \alpha \in \mathbb{C} \quad [4] \tag{12.64}$$

Let $\varepsilon = \pm be$.

$$\int_{S_\gamma} \lambda_p(x)\chi_p(\varepsilon\xi x)d_px \quad [1b)], [15]$$

$$= p^\gamma(1-1/p), \quad |\xi|_p \leqslant p^{-\gamma}, \gamma = 2k. \tag{12.65}$$

$$= 0, \quad |\xi|_p \leqslant p^{-\gamma}, \gamma = 2k+1. \tag{12.66}$$

$$= -p^{\gamma-1}, \quad |\xi|_p = p^{-\gamma+1}, \gamma = 2k. \tag{12.67}$$

$$= \Big(\frac{\xi_0}{p}\Big)p^{\gamma-1/2}, \quad |\xi|_p \leqslant p^{-\gamma+1}, \gamma = 2k+1, p \equiv 1(\mathrm{mod}\,4). \tag{12.68}$$

$$= -\varepsilon\Big(\frac{\xi_0}{p}\Big)p^{\gamma-1/2}, \quad |\xi|_p \leqslant p^{-\gamma+1}, \gamma = 2k+1, p \equiv 3(\mathrm{mod}\,4). \tag{12.69}$$

$$= 0, \quad |\xi|_p \geqslant p^{-\gamma+2}. \tag{12.70}$$

$$\int_{S_\gamma} \lambda_2(x)\chi_2(\varepsilon\xi x)d_2x \quad [10b)], [15]$$

$$= 2^{\gamma-3/2}, \quad |\xi|_2 \leqslant 2^{-\gamma}, \gamma = 2k. \tag{12.71}$$

$$= 0, \quad |\xi|_2 \leqslant 2^{-\gamma}, \gamma = 2k+1. \tag{12.72}$$

$$= -2^{\gamma-3/2}, \quad |\xi|_2 = 2^{-\gamma+1}, \gamma = 2k. \tag{12.73}$$

$$= 0, \quad |\xi|_2 = 2^{-\gamma+1}, \gamma = 2k+1. \tag{12.74}$$

$$= -\varepsilon(-1)^{\xi_1}2^{\gamma-3/2}, \quad |\xi|_2 = 2^{-\gamma+2}, \gamma = 2k. \tag{12.75}$$

$$= 0, \quad |\xi|_2 = 2^{-\gamma+2}, \gamma = 2k+1. \tag{12.76}$$

$$= 0, \quad |\xi|_2 \geqslant 2^{-\gamma+3}, \gamma = 2k. \tag{12.77}$$

$$= i^{\xi_1}(-1)^{\xi_2}2^{\gamma-3}(1+i)(1+i\varepsilon)[1-\varepsilon(-1)^{\xi_1}],$$
$$|\xi|_2 = 2^{-\gamma+3}, \gamma = 2k+1. \tag{12.78}$$

$$= 0, \quad |\xi|_2 \geqslant 2^{-\gamma+4}, \gamma = 2k+1. \tag{12.79}$$

$$\int_{|x|_p \geqslant 1} \lambda_p(x)|x|_p^{\alpha-1}\chi_p(\varepsilon\xi^2 x)d_px = 0, \quad |\xi|_p \geqslant p, p \neq 2. \tag{12.80}$$

$$= (1-1/p)\frac{1-p^{2\alpha}|\xi|_p^{-2\alpha}}{1-p^{2\alpha}} + p^{\alpha-1/2}|\xi|_p^{-2\alpha},$$
$$|\xi|_p \leqslant 1, p \equiv 1(\mathrm{mod}\,4). \tag{12.81}$$

$$= (1-1/p)\frac{1-p^{2\alpha}|\xi|_p^{-2\alpha}}{1-p^{2\alpha}} - \varepsilon p^{\alpha-1/2}|\xi|_p^{-2\alpha},$$
$$|\xi|_p \leqslant 1, p \equiv 3(\mathrm{mod}\,4). \tag{12.82}$$

$$\int_{|x|_p \geqslant 1} \lambda_p(x)|x|_p^{-3/2}\chi_p(-\xi^2 x)d_p x = \Omega(|\xi|_p), \quad p \neq 2 \quad [15]. \tag{12.83}$$

$$\int_{|x|_2 \geqslant 4} \lambda_2(x)|x|_p^{-3/2}\chi_p(-\xi^2 x)d_2 x = \sqrt{2}\Omega(|\xi|_p), \quad p = 2 \quad [15]. \tag{12.84}$$

§13. The Gaussian integrals

The Gaussian integral is called an integral of the form

$$\int f(x)\chi_p(ax^2 + bx)d_p x, \quad a \in \mathbb{Q}_p^\times, b \in \mathbb{Q}_p.$$

Various formulae for the Gaussian integrals are contained in [2a)],[6]–[8], [10b)]. The most full lists of them are collected in [1a)],[1b)]. Here

$$\epsilon = \varepsilon_0 + \varepsilon_1 p + \varepsilon_2 p^2 + \dots .$$

$$\int_{S_\gamma} \chi_p[\epsilon(x-y)^2]d_p y$$

$$= p^\gamma \chi_p(\epsilon x^2)\big[(1-1/p)\Omega(p^\gamma |x|_p) - 1/p\delta(|x|_p - p^{1-\gamma})\big],$$

$$\gamma \leqslant 0, p \neq 2. \tag{13.1}$$

$$= \delta(|x|_p - p^\gamma), \quad \gamma \geqslant 1, p \neq 2. \tag{13.2}$$

$$= 2^{\gamma-1}\chi_2(\epsilon x^2)\big[\Omega(2^{\gamma-1}|x|_2) - \delta(|x|_2 - 2^{2-\gamma})\big], \quad \gamma \leqslant 0, p = 2. \tag{13.3}$$

$$= [\sqrt{2}\lambda_2(\epsilon) - 1]\Omega(|x|_2) + \delta(|x|_2 - 2), \quad \gamma = 1, p = 2. \tag{13.4}$$

$$= \sqrt{2}\lambda_2(\epsilon)\delta(|x|_2 - 2^\gamma), \quad \gamma \geqslant 2, p = 2. \tag{13.5}$$

$$\int_{S_\gamma} \chi_p[\epsilon p(x-y)^2]d_p y$$

$$= p^\gamma \chi_p(\epsilon p x^2)\big[(1-1/p)\Omega(p^{1-\gamma}|x|_p)$$

$$- 1/p\delta(|x|_p - p^{2-\gamma})\big], \quad \gamma \leqslant 0, p \neq 2. \tag{13.6}$$

$$= [\sqrt{p}\lambda_p(\epsilon p) - \chi_p(\epsilon p x^2)]\Omega(|px|_p), \quad \gamma = 1, p \neq 2. \tag{13.7}$$

$$= \sqrt{p}\lambda_p(\epsilon p)\delta(|x|_p - p^\gamma), \quad \gamma \geqslant 2, p \neq 2. \tag{13.8}$$

$$= 2^{\gamma-1}\chi_2(2\epsilon x^2)[\Omega(2^{\gamma-2}|x|_2) - \delta(|x|_2 - 2^{3-\gamma})], \quad \gamma \leqslant 0, p = 2. \tag{13.9}$$

$$= -\Omega(|x|_2) + \delta(|x|_2 - 2) + \lambda_2(2\epsilon)\delta(|x|_2 - 4), \quad \gamma = 1, p = 2. \tag{13.10}$$

$$= 2\lambda_2(2\epsilon)\Omega(|2x|_2), \quad \gamma = 2, p = 2. \tag{13.11}$$

$$= 2\lambda_2(2\epsilon)\delta(|x|_2 - 2^\gamma), \quad \gamma \geqslant 3, p = 2. \tag{13.12}$$

$$\begin{aligned}
&\int_{S_\gamma} \chi_p(ax^2+\xi x)d_px \\
&= \lambda_p(a)|2a|_p^{-1/2}\chi_p\big(-\xi^2/4a\big)\delta\big(|\xi/2a|_p - p^\gamma\big), \\
&|4a|_p \geqslant p^{2-2\gamma}. \qquad (13.13)\\
&= |2a|_p^{-1/2}\Big[\lambda_p(a)\chi_p\big(-\xi^2/4a\big) - \frac{1}{\sqrt{p}}\Big]\Omega(p^{1-\gamma}|\xi|_p), \\
&|a|_p = p^{1-2\gamma}. \qquad (13.14)
\end{aligned}$$

$$\begin{aligned}
&\int_{B_\gamma} \chi_p(ax^2+\xi x)d_px = p^\gamma\Omega(p^\gamma|\xi|_p), \quad |a|_p p^{2\gamma} \leqslant 1. \qquad (13.15)\\
&= \lambda_p(a)|2a|_p^{-1/2}\chi_p\big(-\xi^2/4a\big)\Omega(\big(p^{-\gamma}|\xi/2a|_p\big), \quad |4a|_p p^{2\gamma} > 1. \qquad (13.16)\\
&= 2^\gamma\lambda_2(a)\chi_2\big(-\xi^2/4a\big)\delta(|\xi|_2 - 2^{1-\gamma}), \quad |a|_2 2^{2\gamma} = 2, p=2. \qquad (13.17)\\
&= 2^{\gamma-1/2}\lambda_2(a)\chi_2\big(-\xi^2/4a\big)\Omega(2^\gamma|\xi|_2), \quad |a|_2 2^{2\gamma} = 4, p=2. \qquad (13.18)
\end{aligned}$$

$$\begin{aligned}
&\int \chi_p(ax^2+\xi x)d_px, \quad a \neq 0 \\
&= \lambda_p(a)|2a|_p^{-1/2}\chi_p\big(-\xi^2/4a\big). \qquad (13.19)\\
&= \chi_p(-\xi^2/2), \quad a = 1/2, p \neq 2. \qquad (13.20)\\
&= \exp(i\pi/4)\chi_p(-\xi^2/2), \quad a = 1/2, p = 2. \qquad (13.21)
\end{aligned}$$

$$\begin{aligned}
&\int \exp\,(-|y|_p^2)\chi_p[a(x-y)^2]d_py, \quad a \neq 0, \gamma = \gamma(a) \\
&= |a|_p^{-1/2}S(|a|_p^{-1}, 1/p), \quad |x|_p\sqrt{|a|_p} \leqslant 1, \gamma = 2k, p \neq 2. \qquad (13.22)\\
&= 1/\sqrt{p}|a|_p^{-1/2}S(1/p|a|_p^{-1}, 1/p) + [\lambda_p(a) - 1/\sqrt{p}]|a|_p^{-1/2}\exp(-|pa|_p^{-1}), \\
&|x|_p\sqrt{p|a|_p} \leqslant 1, \gamma = 2k+1, p \neq 2. \qquad (13.23)\\
&= \lambda_p(a)|a|_p^{-1/2}\exp(-|x|_p^2) + |ax|_p^{-1}\chi_p(ax^2)[S(|ax|_p^{-2}, 1/p) \\
&- \exp(-|pax|_p^{-2})], \quad |x|_p\sqrt{|a|_p} \geqslant \sqrt{p}, p \neq 2. \qquad (13.24)\\
&= [\sqrt{2}\lambda_2(a) - 1]|a|_2^{-1/2}\exp(-|4a|_2^{-1}) + |a|_2^{-1/2}S(|a|_2^{-1}, 1/2), \\
&|x|_2\sqrt{|a|_2} \leqslant 1, \gamma = 2k, p = 2. \qquad (13.25)
\end{aligned}$$

$$= |a|_2^{-1/2}\exp(-|4a|_2^{-1}) + [\sqrt{2}\lambda_2(a) - 1]|a|_2^{-1/2}S(|a|_2^{-1}, 1/2),$$

$$|x|_2\sqrt{|a|_2} = 2, \gamma = 2k, p = 2. \tag{13.26}$$

$$= (2|a|_2)^{-1/2}S((2|a|_2)^{-1}, 1/2) - (2|a|_2)^{-1/2}\exp(-|2a|_2^{-1})$$

$$+ \lambda_2(a)|2a|_2^{-1/2}\exp(-|8a|_2^{-1}),$$

$$|x|_2\sqrt{2|a|_2} \leqslant 1, \gamma = 2k+1, p = 2. \tag{13.27}$$

$$= |2a|_2^{-1/2}S(|2a|_2^{-1}, 1/2) + \lambda_2(a)|2a|_2^{-1/2}\exp(-|8a|_2^{-1}),$$

$$|x|_2\sqrt{|a|_2} = \sqrt{2}, \gamma = 2k+1, p = 2. \tag{13.28}$$

$$= \lambda_2(a)(2|a|_2)^{-1/2}S(|2a|_2^{-1}, 1/2),$$

$$|x|_2\sqrt{|a|_2} = 2\sqrt{2}, \gamma = 2k+1, p = 2. \tag{13.29}$$

$$= \lambda_2(a)|2a|_2^{-1/2}\exp(-|x|_2^2) + |2ax|_2^{-1}\chi_2(ax^2)[S(|2ax|_2^{-2}, 1/2)$$

$$- 2\exp(-|4ax|_2^{-2})], \quad |x|_2\sqrt{|a|_2} > 2, p = 2. \tag{13.30}$$

$$\sim \frac{p^4 + p^3}{p^2 + p + 1}|2ax|_p^{-3}\chi_p(ax^2) + O(|x|_p^{-5}), |x|_p \to \infty. \tag{13.31}$$

$$\sim |a|_p^{-1/2}S(|a|_p^{-1}, p^{-1}) + O[|a|_p^{-1/2}\exp(-|p^2a|_p^{-1})],$$

$$|a|_p \to 0, \gamma = 2k. \tag{13.32}$$

$$\sim (p|a|_p)^{-1/2}S((p|a|_p)^{-1}, p^{-1}) + O[|a|_p^{-1/2}\exp(-|pa|_p^{-1})],$$

$$|a|_p \to 0, \gamma = 2k+1. \tag{13.33}$$

Here

$$S(\alpha, q) = (1-q)\sum_{k=0}^{\infty}\frac{(-\alpha)^k}{k!(1-q^{2k+1})}, \quad |q| < 1, \alpha \in \mathbb{C}.$$

This function satisfies the relation

$$S(\alpha q^2, q) = 1/qS(\alpha, q) + (1 - 1/q)e^{-\alpha}. \tag{13.34}$$

§14. Two variables

$$\int_{B_0^2} d_p^2 x = 1. \tag{14.1}$$

$$\int_{B_\gamma^2} d_p^2 x = p^{2\gamma}. \tag{14.2}$$

$$\int_{S^2_\gamma} d^2_p x = (1-p^{-2})p^{2\gamma}. \tag{14.3}$$

$$\int_{B^2_\gamma} f(|x|_p) d^2_p x = (1-p^{-2}) \sum_{k=-\infty}^{\gamma} p^{2k} f(p^k). \tag{14.4}$$

$$\int f(|x|_p) d^2_p x = (1-p^{-2}) \sum_{k=-\infty}^{\infty} p^{2k} f(p^k). \tag{14.5}$$

$$\int_{B^2_\gamma} |x|_p^{\alpha-2} d^2_p x = \frac{1-p^{-2}}{1-p^{-\alpha}} p^{\alpha\gamma}, \quad \operatorname{Re}\alpha > 0. \tag{14.6}$$

$$\int_{S^2_\gamma} |x|_p^{\alpha-2} d^2_p x = (1-p^{-2})p^{\alpha\gamma}. \tag{14.7}$$

$$\int_{B^2_\gamma} |(x,x)|_p^{\alpha-1} \chi_p((\xi,x)) d^2_p x, \quad \operatorname{Re}\alpha > 0, |(\xi,\xi)|_p > p^{-\gamma}$$

$$= \Gamma_p^2(\alpha)|(\xi,\xi)|_p^{-\alpha}, \quad p \equiv 1 (\operatorname{mod} 4) \quad [3a)]. \tag{14.8}$$

$$= \Gamma_p(\alpha)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha}, \quad p \equiv 3 (\operatorname{mod} 4) \quad [3a)]. \tag{14.9}$$

$$\int |(x,x)|_p^{\alpha-1} \chi_p((\xi,x)) d^2_p x, \quad \operatorname{Re}\alpha > 0, (\xi,\xi) \neq 0$$

$$= \Gamma_p^2(\alpha)|(\xi,\xi)|_p^{-\alpha}, \quad p \equiv 1 (\operatorname{mod} 4) \quad [3a)]. \tag{14.10}$$

$$= \Gamma_p(\alpha)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha}, \quad p \equiv 3 (\operatorname{mod} 4) \quad [3a)]. \tag{14.11}$$

$$\int f((x,x)) \chi_p((\xi,x)) d^2_p x, \quad (\xi,\xi) \neq 0$$

$$= |(\xi,\xi)|_p^{-1} \big[(1-p^{-2}) \sum_{\gamma=0}^{\infty} p^{-2\gamma} f(p^{-2\gamma}|(\xi,\xi)|_p^{-1}) - f(p^2|(\xi,\xi)|_p^{-1})\big],$$

$$p \equiv 3 (\operatorname{mod} 4) [1a)]. \tag{14.12}$$

$$= |(\xi,\xi)|_p^{-1} \Big[(1-1/p)^{-2} \sum_{\gamma=0}^{\infty} \Big(\gamma + \frac{p-3}{p-1}\Big) p^{-\gamma} f(p^{-\gamma}|(\xi,\xi)|_p^{-1})$$

$$- 2(1-1/p) f(p|(\xi,\xi)|_p^{-1}) + f(p^2|(\xi,\xi)|_p^{-1})\Big],$$

$$p \equiv 1 (\operatorname{mod} 4) \quad [1a)]. \tag{14.13}$$

$$\int \frac{\chi_p((\xi,x))}{|(x,x)|_p+m^2}d_p^2x, \quad m\neq 0, (\xi,\xi)\neq 0$$

$$= \frac{1-p^{-2}}{p^2+m^2|(\xi,\xi)|_p}\sum_{\gamma=0}^{\infty}\frac{p^2-p^{-2\gamma}}{1+p^\gamma m^2|(\xi,\xi)|_p}$$

$$= \sum_{\gamma=0}^{\infty}\frac{1}{1+p^{2\gamma}m^2|(\xi,\xi)|_p}-\frac{1}{p^2+p^{2\gamma}m^2|(\xi,\xi)|_p},$$

$$p\equiv 3(\mathrm{mod}\,4) \quad [1a)], [3a)]. \tag{14.14}$$

$$= (1-1/p)^2\sum_{\gamma=0}^{\infty}\Big(\gamma+\frac{p-3}{p-1}\Big)\frac{1}{1+p^\gamma m^2|(\xi,\xi)|_p}$$

$$-2(1-1/p)\frac{1}{p+m^2|(\xi,\xi)|_p}+\frac{1}{p^2+m^2|(\xi,\xi)|_p}$$

$$= \sum_{\gamma=0}^{\infty}(\gamma+1)\Big(\frac{1}{1+p^\gamma m^2|(\xi,\xi)|_p}-\frac{2}{p+p^\gamma m^2|(\xi,\xi)|_p}$$

$$+\frac{1}{p^2+p^\gamma m^2|(\xi,\xi)|_p}\Big), \quad p\equiv 1(\mathrm{mod}\,4) \quad [3a)]. \tag{14.15}$$

$$\sim \frac{p^4}{p^2+1}m^{-4}|(\xi,\xi)|_p^{-2}, |(\xi,\xi)|_p\to\infty, \quad \equiv 3(\mathrm{mod}\,4) \quad [1a)], [3a)]. \tag{14.16}$$

$$\sim -\frac{p^4}{(p+1)^2}m^{-4}|(\xi,\xi)|_p^{-2}, |(\xi,\xi)|_p\to\infty, \quad p\equiv 1(\mathrm{mod}\,4) \quad [3a)]. \tag{14.17}$$

$$\int |x|_p^{\alpha-1}|1-x|_p^{\beta-1}|x-y|_p^{\gamma}|y|_p^{\alpha'-1}|1-y|_p^{\beta'-1}d_pxd_py$$

$$= \Gamma_p(\gamma)\int |t|_p^{2-\alpha-\beta-\alpha'-\beta'}B_p(t;\alpha,\beta)B_p(-t;\alpha',\beta')d_pt$$

$$= B_p(\alpha,\beta)B_p(\alpha',\beta')+B_p(\alpha,\beta)B_p(\gamma,\alpha'+\beta'-1)+B_p(\alpha',\beta')B_p(\gamma,\alpha+\beta-1)$$

$$+B_p(\alpha+\beta-1,\alpha'+\beta'-1)B_p(\gamma,3-\alpha-\beta-\alpha'-\beta')-B_p(\alpha,\alpha')B_p(\gamma,\alpha+\alpha')$$

$$-B_p(\beta,\beta')B_p(\gamma,\beta+\beta')+\Gamma_p(\gamma)p^{-\gamma}\Big\{[\Gamma_p(\alpha+\beta-1)p^{1-\alpha-\beta}+B_p(\alpha,\beta)]$$

$$\times[\Gamma_p(\alpha'+\beta'-1)p^{1-\alpha'-\beta'}+B_p(\alpha',\beta')]-[\Gamma_p(\alpha)p^{-\alpha}+\Gamma_p(\beta)p^{-\beta}]$$

$$\times[\Gamma_p(\alpha')p^{-\alpha'}+\Gamma_p(\beta')p^{-\beta'}]\Big\},$$

$$\operatorname{Re}\alpha>0,\operatorname{Re}\beta>0,\operatorname{Re}\gamma>0,\operatorname{Re}\alpha'>0,\operatorname{Re}\beta'>0. \tag{14.18}$$

Here

$$B_p(t;\alpha,\beta)=\int|x|_p^{\alpha-1}|t-x|_p^{\beta-1}\chi_p(x)d_px,\quad B_p(1;\alpha,\beta)=B_p(\alpha,\beta).$$

Below in formulas (14.19)–(14.29) we use the notations for the field $\mathbb{Q}_p(\sqrt{d})$, $d\notin\mathbb{Q}_p^{\times 2}$ (see §9). In particular, (see (9.2) and (9.6))

$$B_\gamma^2=[z\in\mathbb{Q}_p(\sqrt{d}):|z\bar{z}|_p\leqslant q^\gamma];\quad \alpha_k=2k\pi i/\ln q, k\in Z.$$

$$\int_{B_0^2}d_pz=1. \tag{14.19}$$

$$\int_{B_0^2}|z\bar{z}|_p^{\alpha-1}d_pz=\frac{1-q^{-1}}{1-q^{-\alpha}},\quad \operatorname{Re}\alpha>0. \tag{14.20}$$

$$\int_{B_\gamma^2}|z\bar{z}|_p^{\alpha-1}d_pz=\frac{1-q^{-1}}{1-q^{-\alpha}}q^{\alpha\gamma},\quad \operatorname{Re}\alpha>0. \tag{14.21}$$

$$\int_{B_0^2}f(z\bar{z})d_pz=\frac{1}{C_{p,d}}\int_{B_0}f(x)\theta_d^+(x)d_px,\quad p\neq 2\quad [4] \tag{14.22}$$

where quantity $C_{p,d}$ is defined in (9.16) and (9.17).

$$\delta\sqrt{|4d|_p}\int|z\bar{z}|_p^{\alpha-1}\chi_p(z\zeta+\bar{z}\bar{\zeta})d_pz,\quad \operatorname{Re}\alpha>0$$

$$=\Gamma_{p,d}(\alpha)|\zeta\bar{\zeta}|_p^{-\alpha},\quad \zeta\neq 0\quad [2a)]. \tag{14.23}$$

$$=\Gamma_{p,d}(\alpha),\quad \zeta=1. \tag{14.24}$$

$$\delta\sqrt{|4d|_p}\int_{B_\gamma^2}|z\bar{z}|_p^{\alpha-1}\chi_p(z+\bar{z})d_pz=\Gamma_{p,d}(\alpha),$$

$$\operatorname{Re}\alpha>0,\gamma\geqslant 1\quad [2a)]. \tag{14.25}$$

$$\int|\zeta\bar{\zeta}|_p^{\alpha-1}|(z-\zeta)(\bar{z}-\bar{\zeta})|_p^{\beta-1}d_p\zeta,\quad \operatorname{Re}\alpha>0,\operatorname{Re}\beta>0,\operatorname{Re}(\alpha+\beta)<1$$

$$=B_q(\alpha,\beta)|z\bar{z}|_p^{\alpha+\beta-1},\quad z\neq 0\quad [2a)]. \tag{14.26}$$

$$=B_q(\alpha,\beta),\quad z=1. \tag{14.27}$$

$$\int \chi_p(\xi z\bar{z})d_pz, \quad \xi \neq 0, p \neq 2$$

$$= \frac{\operatorname{sgn}_{p,d}\xi}{|\xi|_p}, \quad |d|_p = 1, d \notin \mathbb{Q}_p^{\times 2} \quad [4]. \tag{14.28}$$

$$= \pm\sqrt{p\operatorname{sgn}_{p,d}(-1)}\frac{\operatorname{sgn}_{p,d}\xi}{|\xi|_p}, \quad |d|_p = 1/p \quad [4]. \tag{14.29}$$

§15. n-Variables

$$\int_{B_0^n} d_p^n x = 1. \tag{15.1}$$

$$\int_{S_0^n} d_p^n x = 1 - p^{-n}. \tag{15.2}$$

$$\int_{B_\gamma^n} d_p^n x = p^{n\gamma}. \tag{15.3}$$

$$\int_{S_\gamma^n} d_p^n x = (1 - p^{-n})p^{n\gamma}. \tag{15.4}$$

$$\int_{B_\gamma^n} f(|x|_p)d_p^n x = (1 - p^{-n}) \sum_{k=-\infty}^{\gamma} p^{nk} f(p^k). \tag{15.5}$$

$$\int f(|x|_p)d_p^n x = (1 - p^{-n}) \sum_{k=-\infty}^{\infty} p^{nk} f(p^k). \tag{15.6}$$

$$\int_{B_\gamma^n} |x|_p^{\alpha-n} d_p^n x = \frac{1 - p^{-n}}{1 - p^{-\alpha}} p^{\alpha\gamma}, \quad \operatorname{Re}\alpha > 0. \tag{15.7}$$

$$\int_{S_\gamma^n} |x|_p^{\alpha-n} d_p^n x = (1 - p^{-n})p^{\alpha\gamma}. \tag{15.8}$$

$$\int_{|x|_p > p^\gamma} |x|_p^{\alpha-n} d_p^n x = -\frac{1 - p^{-n}}{1 - p^{-\alpha}} p^{\gamma\alpha}, \operatorname{Re}\alpha < 0. \tag{15.9}$$

$$\int_{S_\gamma^n} \chi_p((\xi, x))d_p^n x$$

$$= (1-p^{-n})p^{\gamma n}\Omega(p^{\gamma}|\xi|_p) - p^{(\gamma-1)n}\delta(|\xi|_p - p^{1-\gamma}) \quad [12], [2a)]. \tag{15.10}$$

$$\int_{B_\gamma^n} \chi_p((\xi, x))d_p^n x = p^{\gamma n}\Omega(p^{\gamma}|\xi|_p) \quad [12], [2a)]. \tag{15.11}$$

$$\int_{B_\gamma^n} |(x,x)|_p^{\alpha-n/2}\chi_p((\xi,x))d_p^n x, \quad |(\xi,\xi)|_p > p^{-\gamma}, \operatorname{Re}\alpha > 0$$

$$= \Gamma_p(\alpha - n/2 + 1)\Gamma_p(\alpha)|(\xi,\xi)|_p^{-\alpha}, \quad n \equiv 0 (\operatorname{mod} 4),$$

$$p \neq 2 \text{ or } n \equiv 2(\operatorname{mod} 4), p \equiv 1(\operatorname{mod} 4) \quad [3b)]. \tag{15.12}$$

$$= (-1)^{\gamma((\xi,\xi))}\Gamma_p(\alpha - n/2 + 1)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$n \equiv 2(\operatorname{mod} 4), p \equiv 3(\operatorname{mod} 4) \quad [3b)]. \tag{15.13}$$

$$\int |(x,x)|_p^{\alpha-n/2}\chi_p((\xi,x))d_p^n, \quad (\xi,\xi) \neq 0, \operatorname{Re}\alpha > 0$$

$$= \Gamma_p(\alpha - n/2 + 1)\Gamma_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$n \equiv 0(\operatorname{mod} 4), p \neq 2 \;\; n \equiv 2(\operatorname{mod} 4), p \equiv 1(\operatorname{mod} 4) \quad [3b)]. \tag{15.14}$$

$$= (-1)^{\gamma((\xi,\xi))}\Gamma_p(\alpha - n/2 + 1)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$n \equiv 2(\operatorname{mod} 4), p \equiv 3(\operatorname{mod} 4) \quad [3b)]. \tag{15.15}$$

$$\int_{B_\gamma^n} |x|_p^{\alpha-n}\chi_p(x_1)d_p^n x = \Gamma_p^{(n)}(\alpha), \quad \operatorname{Re}\alpha > 0, \gamma \geqslant 1. \tag{15.16}$$

$$\int |x|_p^{\alpha-n}\chi_p(x_1)d_p^n x = \Gamma_p^{(n)}(\alpha), \quad \operatorname{Re}\alpha > 0. \tag{15.17}$$

$$\int |x|_p^{\alpha-n}\chi_p((\xi,x))d_p^n x = \Gamma_p^{(n)}(\alpha)|\xi|_p^{-\alpha}, \quad \operatorname{Re}\alpha > 0, \xi \neq 0. \tag{15.18}$$

$$\int |x,m|_p^{\alpha-n}\chi_p((\xi,x))d_p^n x$$

$$= \Gamma_p^{(n)}(\alpha)\big(|\xi|_p^{-\alpha} - |pm|_p^{\alpha}\big)\Omega(|m\xi|_p), \quad m \neq 0 \quad [1a)], [9a)]. \tag{15.19}$$

$$\int |x,1|_p^{-\alpha}\chi_p((\xi,x))d_p^n x$$

$$= \Gamma_p^{(n)}(n-\alpha)(|\xi|_p^{\alpha-n} - p^{\alpha-n})\Omega(|\xi|_p) \equiv J_p^{\alpha}(\xi), \quad \operatorname{Re}\alpha > n. \tag{15.20}$$

$$J_p^n(\xi) = (1-p^{-n})\big(1 - \ln|\xi|_p/\ln p\big)\Omega(|\xi|_p), \quad \alpha = n. \tag{15.21}$$

$$\int J_p^{\alpha}(\xi)J_p^{\beta}(x-\xi)d_p^n\xi = J_p^{\alpha} * J_p^{\beta} = J_p^{\alpha+\beta}, \quad \alpha,\beta \in \mathbb{C}. \tag{15.22}$$

$$\int |x|_p^{\alpha-n}|\varepsilon - x|_p^{\beta-n}d_p^n x = B_p^{(n)}(\alpha,\beta),$$

$$\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}(\alpha+\beta) < n, |\varepsilon|_p = 1. \tag{15.23}$$

$$\int |y|_p^{\alpha-n}|x-y|_p^{\beta-n}d_p^n y = B_p^{(n)}(\alpha,\beta)|x|_p^{\alpha+\beta-n},$$

$$\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0, \operatorname{Re}(\alpha+\beta) < n \quad [1a)], [9a)]. \tag{15.24}$$

$$\int |y,m|_p^{\alpha-n}|x-y,m|_p^{\beta-n}d_p^n y = B_p^{(n)}(\alpha,\beta)|x,m|_p^{\alpha+\beta-n}$$

$$-\Gamma_p^{(n)}(\alpha)|pm|_p^{\alpha}|x,m|_p^{\beta-n} - \Gamma_p^{(n)}(\alpha)|pm|_p^{\beta}|x,m|_p^{\alpha-n},$$

$$\operatorname{Re}(\alpha+\beta) < n, m \neq 0 \quad [9a)]. \tag{15.25}$$

$$\int_{\mathbb{Q}_p^{n-1}} \chi_p\Big\{\sum_{k=0}^{n-1}\Big(\frac{2x_k x_{k+1}}{\sin t_k} - \frac{x_k^2 + x_{k+1}^2}{\operatorname{tg} t_k}\Big)\Big\}d_p x_1 d_p x_2 \dots d_p x_{n-1}$$

$$= \frac{\lambda_p(T_n)}{\sqrt{|T_n|_p}}\prod_{k=0}^{n-1}\frac{\sqrt{|t_k|_p}}{\lambda_p(t_k)}\chi_p\Big(\frac{2x_0 x_n}{\sin T_n} - \frac{x^2 + x_n^2}{\operatorname{tg} T_n}\Big),$$

$$n = 2,3,\dots, p \neq 2, |t_k|_p \leqslant 1/p, k = 0,1,\dots,n-1, T_n = \sum_{k=0}^{n-1} t_k \quad [9a)]. \tag{15.26}$$

Let $x_i \in \mathbb{Q}_p^n, |x_i|_p = 1, i = 1,2,\dots,k < p^n \quad |x_i - x_j|_p = 1,\ i,j = 1,2,\dots, i \neq j$. Denote

$$D_k^n = [x \in \mathbb{Q}_p^n : |x - x_i|_p = 1, i = 1,2,\dots,k].$$

$$\int_{D_k^n} d_p^n x = 1 - kp^{-n}, \quad k \leqslant p^n, p \neq 2 \quad [9b)]. \tag{15.27}$$

Let $G_k^n = [(x_1,x_2,\dots,x_k) \in \mathbb{Q}_p^{kn} : |x_i|_p = 1,\ |x_i - x_j|_p = 1, i,j = 1,2,\dots,k, i \neq j]$.

$$\int_{G_k^n} d_p^n x_1 d_p^n x_2 \dots d_p^n x_k = \prod_{\ell=1}^{k}(1 - \ell p^{-n}) = c_{p,k}^n,$$

$$k \leqslant p^n, p \neq 2 \quad [9b)]. \tag{15.28}$$

Let $x_0 \in \mathbb{Q}_p^n, |x_0|_p = 1 \quad G_k^n(x_0) = [(x_1, \dots, x_k) \in \mathbb{Q}_p^{kn} : |x_i|_p = 1,\ i = 0, 1, \dots, k, \quad |x_i - x_j|_p = 1, i, j = 0, 1, \dots, k, i \neq j]$,

$$\int_{G_k^n(x_0)} d_p^n x_1 d_p^n x_2 \dots d_p^n x_k = \frac{1-(k+1)p^{-n}}{1-p^{-n}} c_{k,p}^n,$$

$$k+1 \leqslant p^n, p \neq 2 \quad [9b)]. \tag{15.29}$$

The Missarov–Lerner integral [12],[17]. Let G be a connected finite graph, $V = V(G)$ and $L = L(G)$ are sets of its vertices and edges respectively. To every line $l \in L$ we associate a complex number a_l, and denote the set $a = \{a_l, l \in L\}$. To every vertex $v \in V$ we associate n-dimensional p-adic vector $x_v = (x_{v1}, x_{v2}, \dots, x_{vn}) \in \mathbb{Q}_p^n$. On the set of vertices V we introduce a hierarchy A by the following way. The hierarchy is a family of subsets of the set V such that: 1)$V \in A$, 2)$v \in A$ for all $v \in V$ and 3)for any pairs $V' \in A, V'' \in A$ either $V' \cap V'' = \vee$ or $V' \in V''$ or $V'' \in V'$. For any $V' \in A, V' \neq V$ we denote by $\theta(V')$ a minimal subset in A containing V' but not coinciding with it. Let $K(V') = [V'' \in A : \theta(V'') = V']$. We consider only such hierarchies A for which

$$1 < |K(V')| \leqslant p^n, V' \in A', \text{ where } A' = [V' \in A : |V'| > 1].$$

Denote

$$a(V') = \sum_{l \in L(G(V'))} a_l, \quad \beta(V') = a(V') + n(|V'| - 1)$$

where $L(G(V'))$ is the set of edges $\{l\}$ of the graph G beginning $i(l)$ and end $f(l)$ of which lay in $V' \subset V = V(G)$. By the condition $\beta(V') > 0, V' \in A'$ the following equality is valid

$$F_G(a) \equiv \int_{Z_p^{n|V|}} \prod_{l \in L} |x_{i(l)} - x_{f(l)}|_p^{a_l} \prod_{v \in V} d_p^n x_v$$

$$= p^{a(V)} \sum_A \prod_{V' \in A'} \frac{1}{p^{\beta(V')} - 1} \frac{(p^n - 1)!}{(p^n - |K(V')|!} \tag{15.30}$$

where the summing is taken over all hierarchies A. (Simbol $|V|$ denotes a number of elements of the set V.) Evaluation of various Feynman integrals is reduced to the integral $F_G(a)$ [12].

§16. Integrals and convolutions of generalized functions

Integral (see §6) of a generalized function $f \in \mathscr{S}'(\mathscr{O})$ on a clopen set $D \in \mathscr{O} \in \mathbb{Q}_p^n$ is called the limit (if it exists!)

$$G\!\!\int_D f(x)d_p^n x = \lim_{k\to\infty}(f\theta_D, \Omega_k).$$

Integrals of generalized functions are contained also in §§12–15 and in §17.

$$G\!\!\int_{B_0^n} d_p^n x = 1. \tag{16.1}$$

$$G\!\!\int_{B_\gamma^n} d_p^n x = p^{\gamma n}. \tag{16.2}$$

$$G\!\!\int_{S_\gamma^n} d_p^n x = (1-p^{-n})p^{\gamma n}. \tag{16.3}$$

$$G\!\!\int f(x)d_p^n x = \int f(x)d_p^n x, \quad f \in \mathscr{L}^1. \tag{16.4}$$

$$G\!\!\int f(x)d_p^n x = \lim_{\gamma\to\infty}\int_{B_\gamma^n} f(x)d_p^n x, \quad f \in \mathscr{L}^1_{\mathrm{loc}}. \tag{16.5}$$

$$G\!\!\int_D f(x)d_p^n x = \int_D f(x)d_p^n x, \quad f \in \mathscr{L}^1(D). \tag{16.6}$$

$$G\!\!\int f(x)d_p^n x = \lim_{\gamma\to\infty}\int_{B_\gamma^n} f(x)d_p^n x, \quad f \in \mathscr{L}^p. \tag{16.7}$$

$$G\!\!\int f(x)d_p^n x = (f, \Omega_N), \quad f \in \mathscr{S}', \mathrm{spt} \in B_N^n. \tag{16.8}$$

$$G\!\!\int_D f(x)d_p^n x = (f, \theta_D), \quad f \in \mathscr{S}'(\mathscr{O}) \tag{16.9}$$

where D is an open compact in $\mathscr{O}$.

$$G\!\!\int f(x)d_p^n x = \lim_{\gamma\to\infty}(f, \Omega_\gamma), \quad f \in \mathscr{S}'. \tag{16.10}$$

$$G\!\!\int \delta(x)d_p^n x = 1. \tag{16.11}$$

$$G\!\!\int_{S_\gamma} \pi(x) d_p x = 0, \quad \pi \not\equiv 1, \alpha \in \mathbb{C} \quad (\text{ cf. } (11.40)). \tag{16.12}$$

$$G\!\!\int_{B_\gamma} |x|_p^{\alpha-1} \pi(x) d_p x = 0, \quad \pi \not\equiv 1, \alpha \in \mathbb{C}. \tag{16.13}$$

$$G\!\!\int |x|_p^{\alpha-1} \pi(x) d_p x = 0, \quad \pi \not\equiv 1, \alpha \in \mathbb{C}. \tag{16.14}$$

$$G\!\!\int_{B_\gamma} |x|_p^{\alpha-1} d_p x = \frac{1-p^{-1}}{1-p^{-\alpha}} p^{\alpha\gamma},$$
$$\alpha \neq \alpha_k, k \in Z \quad (\text{ cf. } (11.18)). \tag{16.15}$$

$$G\!\!\int |x|_p^{\alpha-1} d_p x = 0, \quad \alpha \neq \alpha_k, k \in Z \quad (\text{ cf. } (11.18)). \tag{16.16}$$

$$G\!\!\int_{S_\gamma} |x-a|_p^{\alpha-1} d_p x = \frac{p-2+p^{-\alpha}}{p(1-p^{-\alpha})} |a|_p^\alpha,$$
$$|a|_p = p^\gamma, \alpha \neq \alpha_k, k \in Z \quad (\text{ cf. } (11.20)). \tag{16.17}$$

$$G\!\!\int_{B_\gamma} |x^2+a^2|_p^{(\alpha-1)/2} d_p x = \frac{1-p^{\alpha-1}}{1-p^\alpha} |a|_p^\alpha + \frac{1-p^{-1}}{1-p^{-\alpha}} p^{\alpha\gamma},$$
$$0 \neq |a|_p \leqslant p^\gamma, \alpha \neq \alpha_k, k \in Z, p \equiv 3 (\operatorname{mod} 4) \quad (\text{ cf. } (11.28)). \tag{16.18}$$

$$G\!\!\int |x^2+a^2|_p^{(\alpha-1)/2} d_p x = \frac{1-p^{\alpha-1}}{1-p^\alpha} |a|_p^\alpha,$$
$$a \neq 0, \alpha \neq \alpha_k, k \in Z, p \equiv 3 (\operatorname{mod} 4) \quad (\text{ cf. } (11.30)). \tag{16.19}$$

$$G\!\!\int_{B_\gamma} |x^2+a^2|_p^{\alpha-1} d_p x$$
$$= \Big[1 - 2/p + (1-1/p)\Big(\frac{2}{p^\alpha - 1} + \frac{1}{p^{1-2\alpha}-1}\Big)\Big] |a|_p^{2\alpha-1} - \frac{(1-1/p)p^{(2\alpha-1)\gamma}}{1-p^{2\alpha-1}},$$
$$0 \neq |a|_p \leqslant p^\gamma, \alpha \neq \{\alpha_k, (1-\alpha_k)/2, k \in Z\}, p \equiv 1 (\operatorname{mod} 4). \tag{16.20}$$

$$G\!\!\int |x^2+a^2|_p^{\alpha-1} d_p x$$
$$= \Big[1 - 2/p + (1-1/p)\Big(\frac{2}{p^\alpha - 1} + \frac{1}{p^{1-2\alpha}-1}\Big)\Big] |a|_p^{2\alpha-1},$$
$$\alpha \neq \{\alpha_k, (1-a_k)/2, k \in Z\}, a \neq 0, p \equiv 1 (\operatorname{mod} 4) \quad (\text{ cf. } (11.31)). \tag{16.21}$$

$$G\!\!\int_{S_0} |x^2+1|_p^{\alpha-1} d_p x = 1 - 3/p - 2\frac{1-p^{-1}}{1-p^{\alpha}},$$

$$\alpha \neq \alpha_k, k \in Z, p \equiv 1 (\mathrm{mod}\, 4) \quad (\text{ cf. } (11.32)). \tag{16.22}$$

$$|x|_p^{\alpha-1} * |x|_p^{\beta-1} = B_p(\alpha,\beta)|x|_p^{\alpha+\beta-1},$$

$$(\alpha,\beta) \neq (\alpha_k,\alpha_j), (k,j) \in Z^2. \tag{16.23}$$

$$G\!\!\int |x|_p^{\alpha-1}|1-x|_p^{\beta-1} d_p x = B_p(\alpha,\beta),$$

$$(\alpha,\beta) \neq (\alpha_k,\alpha_j), (k,j) \in Z^2. \tag{16.24}$$

$$|x,m|_p^{\alpha-1} * |x,m|_p^{\beta-1} = B_p(\alpha,\beta)|x,m|_p^{\alpha+\beta-1}$$

$$-\Gamma_p(\alpha)|pm|_p^{\alpha}|x,m|_p^{\beta-1} - \Gamma_p(\beta)|pm|_p^{\beta}|x,m|_p^{\alpha-1},$$

$$m \neq 0, (\alpha,\beta) \neq (\alpha_k,\alpha_j), (k,j) \in Z^2 \quad (\text{ cf. } (11.68)). \tag{16.25}$$

$$G\!\!\int |x,m|_p^{\alpha-1}\chi_p(\xi x) d_p x$$

$$= \Gamma_p(\alpha)(|\xi|_p^{-\alpha} - |pm|_p^{\alpha})\Omega(|m\xi|_p),$$

$$m \neq 0, \alpha \in \mathbb{C} \quad (\text{ see } (12.48)). \tag{16.26}$$

$$G\!\!\int_{S_\gamma} \delta(x_0-k) d_p x = p^{\gamma-1}, \quad k = 1,2,\dots,p-1 \quad (\text{ see } (11.33)). \tag{16.27}$$

$$G\!\!\int_{S_\gamma} [1-\delta(x_0-k)] d_p x = (1-2/p)p^{\gamma},$$

$$k = 1,2,\dots,p-1 \quad (\text{ see } (11.34)). \tag{16.28}$$

$$G\!\!\int_{S_\gamma} \delta(x_n-k) d_p x = (1-1/p)p^{\gamma-1},$$

$$k = 0,1,\dots,p-1, n \in Z_+ \quad (\text{ see } (11.35)). \tag{16.29}$$

$$G\!\!\int_{S_\gamma} [1-\delta(x_n-k)] d_p x = (1-1/p)^2 p^{\gamma},$$

$$k = 0,1,\dots,p-1, n \in Z_+ \quad (\text{ see } (11.36)). \tag{16.30}$$

$$G\!\!\int_{S_\gamma} \delta(x_0-k_0)\prod_{l=1}^{n}\delta(x_l-k_l) = p^{\gamma-n-1},$$

$$k_l = 0, 1, \dots, p-1, k_0 \neq 0, n = 0, 1, \dots \quad (\text{ see } (11.37)). \tag{16.31}$$

$$G\!\!\int_{S_\gamma} \Big[1 - \delta(x_0 - k_0) \prod_{l=1}^{n} \delta(x_l - k_l)\Big] = (1 - p^{-1} - p^{-n-1}) p^{\gamma},$$

$$k_l = 0, 1, \dots, p-1, k_0 \neq 0, n = 0, 1, \dots \quad (\text{ see } (11.38)). \tag{16.32}$$

$$G\!\!\int_{S_\gamma} \Big(\prod_{l=1}^{n} \delta(x_l - k_l)\Big) d_p x = (1 - 1/p) p^{\gamma - n},$$

$$k_l = 0, 1, \dots, p-1, n \in Z_+. \tag{16.33}$$

$$G\!\!\int_{S_\gamma} \Big[1 - \prod_{l=1}^{n} \delta(x_{i_l} - k_{i_l})\Big] d_p x = (1 - 1/p)(1 - p^{-n}) p^{\gamma},$$

$$k_l = 0, 1, \dots, p-1, n \in Z_+. \tag{16.34}$$

$$G\!\!\int_{S_\gamma^n} |x|_p^{\alpha - n} d_p^n x = (1 - p^{-n}) p^{\alpha\gamma}, \quad \alpha \in \mathbb{C} \quad (\text{ see } (15.8)). \tag{16.35}$$

$$G\!\!\int_{B_\gamma^n} |x|_p^{\alpha - n} d_p^n x = \frac{1 - p^{-n}}{1 - p^{-\alpha}} p^{\alpha\gamma}, \quad \alpha \neq \alpha_k, k \in Z \quad (\text{ cf. } (15.7)). \tag{16.36}$$

$$G\!\!\int |x|_p^{\alpha - n} d_p^n x = 0, \quad \alpha \neq \alpha_k, k \in Z, n \in Z_+. \tag{16.37}$$

$$G\!\!\int_{B_\gamma^n} |(x,x)|_p^{\alpha - n/2} \chi_p((\xi, x)) d_p^n x, \quad |(\xi,\xi)|_p > p^{\gamma}$$

$$= \Gamma_p(\alpha - n/2 + 1)\Gamma_p(\alpha) |(\xi,\xi)|_p^{-\alpha}, \quad \alpha \neq \{\alpha_k, \alpha_k + n/2 - 1, k \in Z\},$$

$$n \equiv 0 (\operatorname{mod} 4), p \neq 2 \text{ or } n \equiv 2 (\operatorname{mod} 4), p \equiv 1 (\operatorname{mod} 4). \tag{16.38}$$

$$= (-1)^{\gamma((\xi,\xi))} \Gamma_p(\alpha - n/2 + 1) \tilde{\Gamma}_p(\alpha) |(\xi,\xi)|_p^{-\alpha},$$

$$\alpha \neq \{\alpha_k - \pi i/\ln p, \alpha_k + n/2 - 1, k \in Z\},$$

$$n \equiv 2 (\operatorname{mod} 4), p \equiv 3 (\operatorname{mod} 4) \quad (\text{ cf. } (15.13)). \tag{16.39}$$

$$= \Gamma_p^2(\alpha) |(\xi,\xi)|_p^{-\alpha}, \quad \alpha \neq \alpha_k, k \in Z, n = 2, p \equiv 1 (\operatorname{mod} 4). \tag{16.40}$$

$$= \Gamma_p(\alpha) \tilde{\Gamma}_p(\alpha) |(\xi,\xi)|_p^{-\alpha}, \quad \alpha \neq \{\alpha_k, \alpha_k - \pi i/\ln p, k \in Z\},$$

$$n = 2, p \equiv 3 (\operatorname{mod} 4) \quad (\text{ see } (14.9)). \tag{16.41}$$

$$G\!\!\int |(x,x)|_p^{\alpha-n/2}\chi_p((\xi,x))d_p^n x, \quad (\xi,\xi)\neq 0$$
$$= \Gamma_p(\alpha-n/2+1)\Gamma_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$
$$\alpha\neq\{\alpha_k, \alpha_k+n/2-1, k\in Z\},$$
$$n\equiv 0(\mathrm{mod}\,4), p\neq 2 \quad \text{or } n\equiv 2(\mathrm{mod}\,4), p\equiv 1(\mathrm{mod}\,4). \tag{16.42}$$
$$= (-1)^{\gamma((\xi,\xi))}\Gamma_p(\alpha-n/2+1)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$
$$\alpha\neq\{\alpha_k-\pi i/\ln p, \alpha_k+n/2-1, k\in Z\},$$
$$n\equiv 2(\mathrm{mod}\,4), p\equiv 3(\mathrm{mod}\,4) \quad (\text{ see } (15.15)). \tag{16.43}$$
$$= \Gamma_p^2(\alpha)|(\xi,\xi)|_p^{-\alpha}, \quad \alpha\neq\alpha_k, k\in Z,$$
$$n=2, p\equiv 1(\mathrm{mod}\,4) \quad (\text{ cf. } (14.10)). \tag{16.44}$$
$$= \Gamma_p(\alpha)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha}, \alpha\neq\{\alpha_k, \alpha_k-\pi i/\ln p, k\in Z\},$$
$$n=2, p\equiv 3(\mathrm{mod}\,4)(\text{ cf. } (14.11)). \tag{16.45}$$

$$G\!\!\int_{B_\gamma^n} |x|_p^{\alpha-n}\chi_p(x_1)d_p^n x = \Gamma_p^{(n)}(\alpha), \quad \alpha\neq\alpha_k, k\in Z, \gamma\in Z_+. \tag{16.46}$$

$$G\!\!\int |x|_p^{\alpha-n}\chi_p(x_1)d_p^n x = \Gamma_p^{(n)}(\alpha), \quad \alpha\neq\alpha_k, k\in Z. \tag{16.47}$$

$$G\!\!\int |x|_p^{\alpha-n}\chi_p\big((\xi,x)\big)d_p^n x = \Gamma_p^{(n)}(\alpha)|\xi|_p^{-\alpha},$$
$$\alpha\neq\alpha_k, k\in Z, \xi\neq 0 \quad (\text{ cf. } (15.18)). \tag{16.48}$$
$$|x|_p^{\alpha-n} * |x|_p^{\beta-n} = B_p^{(n)}(\alpha,\beta)|x|_p^{\alpha+\beta-n},$$
$$(\alpha,\beta)\neq(\alpha_k,\alpha_j), (k,j)\in Z^2. \tag{16.49}$$

$$G\!\!\int |x,m|_p^{\alpha-n}\chi_p\big((\xi,x)\big)d_p^n x = \Gamma_p^{(n)}(\alpha)\big(|\xi|_p^{-\alpha}-|pm|_p^{\alpha}\big)$$
$$\times\Omega(|m\xi|_p), \quad m\neq 0, \alpha\in\mathbb{C} \quad (\text{ cf. } (15.19)). \tag{16.50}$$
$$G\!\!\int |x,1|_p^{-\alpha}\chi_p\big((\xi,x)\big)d_p^n x$$
$$= \Gamma_p^{(n)}(n-\alpha)\big(|\xi|_p^{\alpha-n}-p^{\alpha-n}\big)\Omega(|\xi|_p), \quad \alpha\in\mathbb{C}. \tag{16.51}$$
$$|x,m|_p^{\alpha-n} * |x,m|_p^{\beta-n} = B_p^{(n)}(\alpha,\beta)|x,m|_p^{\alpha+\beta-n}$$
$$= -\Gamma_p^{(n)}(\alpha)|pm|_p^{\alpha}|x,m|_p^{\beta-n} - \Gamma_p^{(n)}(\beta)|pm|_p^{\beta}|x,m|_p^{\alpha-n},$$

$$(\alpha,\beta)\neq(\alpha_k,\alpha_j),(k,j)\in Z^2, m\neq 0 \quad (\text{ cf. } (15.25)). \tag{16.52}$$

$$(D^\alpha\varphi)(x)=(f_{-\alpha}*\varphi)(x),\quad \varphi\in\mathscr{S}$$

$$=\Gamma_p^{-1}(-\alpha)\int\frac{\varphi(y)-\varphi(x)}{|x-y|_p^{\alpha+1}}d_py,\quad \operatorname{Re}\alpha>0. \tag{16.53}$$

$$=(1-p^{-\alpha-1})^{-1}\int[\varphi(x+y)-\varphi(x+y/p)]|y|_p^{-\alpha-1}d_py,$$

$$\alpha\neq\alpha_k-1,k\in Z. \tag{16.54}$$

$$=-\frac{p-1}{p\ln p}\int\varphi(y)\ln|x-y|_pd_py,\quad \int f(y)d_py=0$$

$$\alpha=\alpha_k-1,k\in Z. \tag{16.55}$$

$$=\varphi(x),\quad \alpha=\alpha_k,k\in Z. \tag{16.56}$$

$$=\int|\xi|_p^\alpha\tilde{\varphi}(\xi)\chi_p(-\xi x)d_p\xi,\quad \operatorname{Re}\alpha>-1. \tag{16.57}$$

$$=\int|\xi|_p^\alpha[\tilde{\varphi}(\xi)\chi_p(-\xi x)-\tilde{\varphi}(0)]d_p\xi,\quad \operatorname{Re}\alpha<-1. \tag{16.58}$$

$$=\int_{Z_p}|\xi|_p^{-1}[\tilde{\varphi}(\xi)\chi_p(-\xi x)-\tilde{\varphi}(0)]d_p\xi+1/p\tilde{\varphi}(0)$$

$$+\int_{\mathbb{Q}_p\setminus Z_p}|\xi|_p^{-1}\tilde{\varphi}(\xi)\chi_p(-\xi x)d_p\xi,\quad \alpha=\alpha_k-1,k\in Z. \tag{16.59}$$

$$D^\alpha\chi_p(ax)=|a|_p^\alpha\chi_p(ax),\quad \alpha\in\mathbb{C},a\neq 0. \tag{16.60}$$

$$D^\alpha\Phi(x)=p^{\gamma\alpha}\Phi(x),\quad \alpha\in\mathbb{R}\quad [1a)]. \tag{16.61}$$

$\Phi(x)=F[\delta(|\xi|_p-p^\gamma)f(\xi)], f\in\mathscr{S}$.

$$D^\alpha[\delta(|x|_p-p^\gamma)\chi_p(ax^2)]$$

$$=p^{\gamma\alpha}|2a|_p^\alpha\delta(|x|_p-p^\gamma)\chi_p(ax^2),\quad \alpha\in\mathbb{R},|2a|_p\leqslant p^{2-2\gamma}\quad [1a)]. \tag{16.62}$$

$$D^\alpha[\eta(x_0)\delta(|x|_p-p^\gamma)]=p^{\alpha(1-\gamma)}\eta(x_0)\delta(|x|_p-p^\gamma),$$

$$\alpha\in\mathbb{R},p\neq 2,\sum_{k=1}^{p-1}\eta(k)=0\quad [2b)]. \tag{16.63}$$

$$(D^\alpha f)(x),\quad f\in\mathscr{S},\operatorname{spt} f\in B_N,|x|_p>p^N$$

$$=\Gamma_p^{-1}(\alpha)|x|_p^{\alpha-1}(f,\Omega_N),\quad \alpha\neq -1\quad [2a)]. \tag{16.64}$$

$$=-\frac{p-1}{p\ln p}\ln|x|_p(f,\Omega_N),\quad \alpha=-1\quad [2a)]. \tag{16.65}$$

$$D^{\alpha}1 = 0, \quad \alpha > 0. \tag{16.66}$$

$$D^{\alpha}\delta(x-a) = f_{-\alpha}(x-a), \quad \alpha \in \mathbb{C}, a \in \mathbb{Q}_p. \tag{16.67}$$

$$\begin{gathered} D^{\alpha}[\delta(|x|_p - p^{\ell-N})\delta(x_0 - j)\chi_p(\epsilon_\ell p^{\ell-2N}x^2)] \\ = p^{\alpha N}\delta(|x|_p - p^{\ell-N})\delta(x_0 - j)\chi_p(\epsilon_\ell p^{\ell-2N}x^2), \\ N \in Z, p \neq 2, \alpha > 0, \ell = 2, 3, \dots, j = 1, 2, \dots, p-1, \\ \epsilon_\ell = \varepsilon_0 + \varepsilon_1 p + \ldots + \varepsilon_{\ell-2}, \\ \varepsilon_s = 0, 1, \dots, p-1, \varepsilon_0 \neq 0, s = 0, 1, \dots, \ell-2 \quad [2c)]. \end{gathered} \tag{16.68}$$

$$\begin{gathered} D^{\alpha}[\Omega(p^{N-1}|x|_p)\chi_p(jp^{-N}x)] = p^{\alpha N}\Omega(p^{N-1}|x|_p)\chi_p(jp^{-N}x), \\ N \in Z, p \neq 2, \alpha > 0, j = 1, 2, \dots, p-1 \quad [2b)]. \end{gathered} \tag{16.69}$$

$$\begin{gathered} D^{\alpha}[\delta(|x|_2 - 2^{\ell+1-N})\chi_2(\epsilon_\ell 2^{\ell-2N}x^2 + 2^{\ell-N-j}x)] \\ = 2^{\alpha N}\delta(|x|_2 - 2^{\ell+1-N})\chi_2(\epsilon_\ell 2^{\ell-2N}x^2 + 2^{\ell-N-j}x), \\ N \in Z, p = 2, \alpha > 0, \ell = 2, 3, \dots, j = 0, 1, \epsilon_\ell = 1 + \varepsilon_1 2 + \ldots + \varepsilon_{\ell-2}2^{\ell-2}, \\ \varepsilon_s = 0, 1, s = 1, 2, \dots, \ell-2 \quad [2b)]. \end{gathered} \tag{16.70}$$

$$\begin{gathered} D^{\alpha}[\Omega(2^N|x - j2^{N-2}|_2) - \delta(|x - j2^{N-2}|_2 - 2^{1-N})] \\ = 2^{\alpha N}[\Omega(2^N|x - j2^{N-2}|_2) - \delta(|x - j2^{N-2}|_2 - 2^{1-N})], \\ N \in Z, p = 2, \alpha > 0, j = 0, 1 \quad [2b)]. \end{gathered} \tag{16.71}$$

$$D^{\alpha}\Omega(p^{-\gamma}|x|_p) = \frac{p-1}{p^{\alpha+1}-1}p^{\alpha(1-\gamma)}, \quad x \in B_\gamma, \alpha > 0 \quad [2c)]. \tag{16.72}$$

$$D^{\alpha}\delta(|x|_p - p^{\gamma}) = \frac{p^{\alpha}+p-2}{p^{\alpha+1}-1}p^{\alpha(1-\gamma)}, \quad x \in S_\gamma, \alpha > 0 \quad [2c)]. \tag{16.73}$$

Let $\mathscr{K}(t,\tau)$ be a real symmetric kernel

$$\mathscr{K}(t,t) = 0, \quad \mathscr{K}(t,\tau) = \rho(1-1/p)^{-1}t^{-\alpha-1}, \tau < t,$$

$$\sigma = \frac{p^{\alpha}+p-2}{p^{\alpha+1}-1}p^{\alpha}, \rho = -\Gamma_p^{-1}(-\alpha)(1-1/p), \sigma + \rho = p^{\alpha}$$

and a function $f \in \mathscr{L}^1_{\text{loc}}$ such that

$$\int_{|x|_p > 1} |f(x)||x|_p^{-\alpha-1} d_p x < \infty.$$

Then

$$(D^{\alpha}f)(x) = -\int \mathscr{K}(|x|_p,|y|_p)f(y)d_py + \sigma|x|_p^{-\alpha}f(x), \quad \alpha>0 \quad [2c)]. \tag{16.74}$$

In the following formulas §16 all integrals

$$G\!\!\int f(z,\bar z)d_pz$$

are understood on the normalized measure $d_pz = \delta^{-1}d_pxd_py, \quad z = x+\sqrt{d}y, \bar z = x-\sqrt{d}y$ of the field $\mathbb{Q}_p(\sqrt{d})$, $d\notin\mathbb{Q}_p^{\times 2}$ (see (9.2)). In particular, $B_\gamma^2 = [z\in\mathbb{Q}_p(\sqrt{d}) : |z\bar z|_p \leqslant q^\gamma]$; $\alpha_k = \frac{2k\pi i}{\ln q}, k\in Z$ (see (9.6)).

$$G\!\!\int_{B_0^2} d_pz = 1. \tag{16.75}$$

$$G\!\!\int_{B_0^2} |z\bar z|_p^{\alpha-1}d_pz = \frac{1-q^{-1}}{1-q^{-\alpha}}, \quad \alpha\neq\alpha_k, k\in Z. \tag{16.76}$$

$$G\!\!\int |z\bar z|_p^{\alpha-1}d_pz = 0, \quad \alpha\neq\alpha_k, k\in Z. \tag{16.77}$$

$$G\!\!\int |z\bar z|_p^{\alpha-1}\chi_p(z+\bar z)d_pz = \frac{\Gamma_{p,d}(\alpha)}{\delta\sqrt{|4d|_p}}, \quad \alpha\neq\alpha_k, k\in Z. \tag{16.78}$$

$$G\!\!\int_{B_1^2} |z\bar z|_p^{\alpha-1}\chi_p(z+\bar z)d_pz = \frac{\Gamma_{p,d}(\alpha)}{\delta\sqrt{|4d|_p}}, \quad \alpha\neq\alpha_k, k\in Z. \tag{16.79}$$

$$|z\bar z|_p^{\alpha-1} * |z\bar z|_p^{\beta-1} = B_q(\alpha,\beta)|z\bar z|_p^{\alpha+\beta-1},$$

$$(\alpha,\beta)\neq(\alpha_k,\alpha_j), (k,j)\in Z^2. \tag{16.80}$$

$$G\!\!\int |z\bar z|_p^{\alpha-1}|(1-z)(1-\bar z)|_p^{\beta-1}d_pz = B_q(\alpha,\beta),$$

$$(\alpha,\beta)\neq(\alpha_k,\alpha_j), (k,j)\in Z^2. \tag{16.81}$$

$$G\!\!\int \chi_p(\xi z\bar z)d_pz = \frac{\operatorname{sgn}_{p,d}\xi}{|\xi|_p} + \frac{1+p}{2p}\delta(\xi),$$

$$p\neq 2, |d|_p = 1, d\notin\mathbb{Q}_p^{\times 2} \quad [4]. \tag{16.82}$$

$$G\!\!\int \chi_p(\xi z\bar{z})d_pz = \pm\sqrt{p\mathrm{sgn}_{p,d}(-1)}\frac{\mathrm{sgn}_{p,d}\xi}{|\xi|_p} + \delta(\xi),$$
$$p \neq 2, |d|_p = 1/p \quad [4]. \tag{16.83}$$

§17. Table of the Fourier transforms

For one-to-one correspondence between *preimage* $f \in \mathscr{S}'$ and its *image* $\tilde{f} \in \mathscr{S}'$ – the Fourier transform of f – we shall use the notation (see §7)

$$f(x) \iff \tilde{f}(\xi).$$

$$\omega_\gamma(x) \iff \delta_\gamma(\xi). \tag{17.1}$$

$$\delta(x) \iff 1(\xi). \tag{17.2}$$

$$f(Ax+b) \iff |\det A|_p^{-1}\chi_p\big(-(A^{-1}b,\xi)\big)\tilde{f}(\bar{A}'\xi),$$
$$\det A \neq 0, b \in \mathbb{Q}_p^n. \tag{17.3}$$

$$f(x-b) \iff \chi_p((b,\xi))\tilde{f}(\xi), \quad b \in \mathbb{Q}_p^n. \tag{17.4}$$

$$\breve{f}(x) \iff \breve{\tilde{f}}(\xi). \tag{17.5}$$

$$f(x) \iff \int f(x)\chi_p((\xi,x))d_p^nx, \quad f \in \mathscr{L}^1. \tag{17.6}$$

$$f(x) \iff \lim_{k\to\infty}\int_{B_k^n} f(x)\chi_p((\xi,x))d_p^nx \text{ in } \mathscr{S}', \quad f \in \mathscr{L}^1_{\mathrm{loc}}. \tag{17.7}$$

$$f(x) \iff \lim_{k\to\infty}\int_{B_k^n} f(x)\chi_p((\xi,x))d_p^nx \text{ in } \mathscr{L}^2, \quad f \in \mathscr{L}^2. \tag{17.8}$$

$$f(x) \iff \big(f(x), \Omega_N(x)\chi_p((\xi,x))\big), \quad \mathrm{spt}\, f \in B_N. \tag{17.9}$$

$$f * g \iff \tilde{f}\cdot\tilde{g}. \tag{17.10}$$

$$f \cdot g \iff \tilde{f} * \tilde{g}. \tag{17.11}$$

$$\delta(|x|_p - p^\gamma) \iff (1-1/p)p^\gamma\Omega(p^\gamma|\xi|_p) - p^{\gamma-1}\delta(|\xi|_p - p^{1-\gamma}). \tag{17.12}$$

$$f(|x|_p)\Omega_\gamma(|x|_p) \iff (1-1/p)\sum_{k=-\infty}^{\gamma} p^k f(p^k)\Omega(p^\gamma|\xi|_p)$$

$$+|\xi|_p^{-1}\Big[(1-1/p)\sum_{k=0}^{\infty}p^{-\gamma}f(p^{-\gamma}|\xi|_p^{-1})-f(p|\xi|_p^{-1})\Big][1-\Omega(p^{\gamma}|\xi|_p)]. \quad (17.13)$$

$$f(|x|_p) \iff |\xi|_p^{-1}\Big[(1-1/p)\sum_{k=0}^{\infty}p^{-\gamma}f(p^{-\gamma}|\xi|_p^{-1})-f(p|\xi|_p^{-1})\Big]. \quad (17.14)$$

$$|x|_p^{\alpha-1} \iff \Gamma_p(\alpha)|\xi|_p^{-\alpha}, \quad \alpha\neq\alpha_k, k\in Z. \quad (17.15)$$

$$\ln|x|_p \iff -(1-1/p)^{-1}\ln p\big(\mathrm{reg}\,|\xi|_p^{-1}+1/p\delta(\xi)\big). \quad (17.16)$$

$$\frac{1}{|x|_p^2+m^2} \iff (1-1/p)\frac{|\xi|_p}{p^2+m^2|\xi|_p^2}$$

$$\times\sum_{\gamma=0}^{\infty}p^{-\gamma}\frac{p^2-p^{-2\gamma}}{p^{-2\gamma}+m^2|\xi|_p^2}, \quad m\neq 0. \quad (17.17)$$

$$|x|_p^{\alpha-1}|1-x|_p^{\beta-1} \iff \big[\Gamma_p(\alpha+\beta-1)|\xi|_p^{1-\alpha-\beta}+B_p(\alpha,\beta)\big]\Omega(|\xi|_p)$$

$$+\big[\Gamma_p(\alpha)|\xi|_p^{-\alpha}+\Gamma_p(\beta)|\xi|_p^{-\beta}\chi_p(\xi)\big][1-\Omega(|\xi|_p)],$$

$$(\alpha,\beta)\neq(\alpha_k,\alpha_j),(k,j)\in Z^2. \quad (17.18)$$

$$\delta(|x|_p-1)|1-x|_p^{\alpha-1} \iff \Gamma_p(\alpha)\chi_p(\xi)|\xi|_p^{-\alpha},$$

$$\gamma(\xi)\geqslant 2, \alpha\neq\alpha_k, k\in Z. \quad (17.19)$$

$$\eta_{x_0}\delta(|x|_p-p^{\gamma}) \iff p^{\gamma-1}\eta'_{\xi_0}\delta(|\xi|_p-p^{1-\gamma}), \quad p\neq 2 \quad (17.20)$$

where

$$\sum_{k=1}^{p-1}\eta_k=0, \quad \eta'_j=\sum_{k=1}^{p-1}\eta_k\exp\big(2\pi i\frac{kj}{p}\big).$$

$$|x|_p^{\alpha-1}\Omega(p^{-\gamma}|x|_p) \iff \frac{1-p^{-1}}{1-p^{-\alpha}}p^{\alpha\gamma}\Omega(p^{\gamma}|\xi|_p)$$

$$+\Gamma_p(\alpha)|\xi|_p^{-\alpha}\big[1-\Omega(p^{\gamma}|\xi|_p)\big], \quad \alpha\neq\alpha_k, k\in Z. \quad (17.21)$$

$$\delta(|x|_p-1)\delta(x_0-p+1) \iff p^{-1}\chi_p(-\xi)\Omega(|p\xi|_p). \quad (17.22)$$

$$\chi_p(x)\Omega(|px|_p) \iff p\delta(|\xi|_p-1)\delta(\xi_0-p+1). \quad (17.23)$$

$$|x,m|_p^{\alpha-1} \iff \Gamma_p(\alpha)\big(|\xi|_p^{-\alpha}-|pm|_p^{\alpha}\big)\Omega(|m\xi|_p), \quad m\neq 0, \alpha\in\mathbb{C}. \quad (17.24)$$

$$f((x,x)) \iff |(\xi,\xi)|_p^{-1}\Big[(1-p^{-2})\sum_{\gamma=0}^{\infty}p^{-2\gamma}f(p^{-2\gamma}|(\xi,\xi)|_p^{-1})$$

$$-f(p^2|(\xi,\xi)|_p^{-1})\Big], \quad n=2, p\equiv 3(\mathrm{mod}\,4). \tag{17.25}$$

$$f((x,x)) \iff |(\xi,\xi)|_p^{-1}\Big[(1-1/p)^2\sum_{\gamma=0}^{\infty}\Big(\gamma+\frac{p-3}{p-1}\Big)p^{-\gamma}f(p^{-\gamma}|(\xi,\xi)|_p^{-1})$$

$$-2(1-1/p)f(p|(\xi,\xi)|_p^{-1})+f(p^2|(\xi,\xi)|_p^{-1})\Big], \quad n=2, p\equiv 1(\mathrm{mod}\,4). \tag{17.26}$$

$$|(x,x)|_p^{\alpha-1} \iff \Gamma_p^2(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$n=2, \alpha\neq\alpha_k, k\in Z, p\equiv 1(\mathrm{mod}\,4). \tag{17.27}$$

$$|(x,x)|_p^{\alpha-1} \iff \Gamma_p(\alpha)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$\alpha\neq\{\alpha_k, \alpha_k-\pi i/\ln p, k\in Z\}, n=2, p\equiv 3(\mathrm{mod}\,4). \tag{17.28}$$

$$\frac{1}{|(x,x)|_p+m^2} \iff \frac{1-p^{-2}}{p^2+m^2|(\xi,\xi)|_p}\sum_{\gamma=0}^{\infty}\frac{p^2-p^{-2\gamma}}{1+p^{\gamma}m^2|(\xi,\xi)|_p},$$

$$n=2, m\neq 0, p\equiv 3(\mathrm{mod}\,4). \tag{17.29}$$

$$\frac{1}{|(x,x)|_p+m^2} \iff (1-1/p)^2\sum_{\gamma=0}^{\infty}\Big(\gamma+\frac{p-3}{p-1}\Big)\frac{1}{1+p^{\gamma}m^2|(\xi,\xi)|_p}$$

$$-2\frac{1-1/p}{p+m^2|(\xi,\xi)|_p}+\frac{1}{p^2+m^2|(\xi,\xi)|_p},$$

$$n=2, m\neq 0, p\equiv 1(\mathrm{mod}\,4). \tag{17.30}$$

$$|(x,x)|_p^{\alpha-n/2} \iff \Gamma_p(\alpha-n/2+1)\Gamma_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$\alpha\neq\{\alpha_k, a_k+n/2-1, k\in Z\}, n\equiv 0(\mathrm{mod}\,4), p\neq 2$$

$$\text{or } n\equiv 2(\mathrm{mod}\,4), p\equiv 1(\mathrm{mod}\,4). \tag{17.31}$$

$$|(x,x)|_p^{\alpha-n/2} \iff (-1)^{\gamma((\xi,\xi))}\Gamma_p(\alpha-n/2+1)\tilde{\Gamma}_p(\alpha)|(\xi,\xi)|_p^{-\alpha},$$

$$\alpha\neq\{\alpha_k-\pi i/\ln p, \alpha_k+n/2-1, k\in Z\},$$

$$n\equiv 2(\mathrm{mod}\,4), n\geqslant 6, p\equiv 3(\mathrm{mod}\,4). \tag{17.32}$$

$$|x|_p^{\alpha-n} \iff \Gamma_p^{(n)}(\alpha)|\xi|_p^{-\alpha}, \quad \alpha\neq\alpha_k, k\in Z. \tag{17.33}$$

$$|x,m|_p^{\alpha-n} \iff \Gamma_p^{(n)}(\alpha)\big(|\xi|_p^{-\alpha} - |pm|_p^{\alpha}\big)\Omega(|m\xi|_p), \quad m \neq 0, \alpha \in \mathbb{C}. \tag{17.34}$$

$$\sqrt{|2a|_p}\chi_p(ax^2)\delta(|x|_p - p^{\gamma}) \iff \lambda_p(a)\chi_p(-\xi^2/4a)$$
$$\times\delta(|\xi|_p - |2a|_p p^{\gamma}), \quad |4a|_p \geqslant p^{2-2\gamma}. \tag{17.35}$$

$$\sqrt{|2a|_p}\chi_p(ax^2)\delta(|x|_p - p^{\gamma}) \iff \big[\lambda_p(a)\chi_p(-\xi^2/4a) - 1/\sqrt{p}\big]$$
$$\times\Omega(p^{1-\gamma}|\xi|_p), \quad p \neq 2, |a|_p = p^{1-2\gamma}. \tag{17.36}$$

$$\chi_p(ax^2)\Omega(p^{-\gamma}|x|_p) \iff p^{\gamma}\Omega(p^{\gamma}|\xi|_p), \quad |a|_p p^{2\gamma} \leqslant 1. \tag{17.37}$$

$$\sqrt{|2a|_p}\chi_p(ax^2)\Omega(p^{-\gamma}|x|_p) \iff \lambda_p(a)\chi_p\big(-\xi^2/4a\big)$$
$$\times\Omega\big(p^{-\gamma}|2a|_p^{-1}|\xi|_p\big), \quad |4a|_p p^{2\gamma} \geqslant p. \tag{17.38}$$

$$\sqrt{|2a|_2}\chi_2(ax^2)\Omega(2^{-\gamma}|x|_2) \iff \lambda_2(a)\chi_2\big(-\xi^2/4a\big)$$
$$\times\delta(|\xi|_2 - 2^{1-\gamma}), \quad p = 2, |a|_2 2^{2\gamma} = 2. \tag{17.39}$$

$$\sqrt{|2a|_2}\chi_2(ax^2)\Omega(2^{-\gamma}|x|_2) \iff \lambda_2(a)\chi_2\big(-\xi^2/4a\big)\Omega(2^{\gamma}|\xi|_2),$$
$$p = 2, |a|_2 2^{2\gamma} = 4. \tag{17.40}$$

$$\chi_p(ax^2) \iff \lambda_p(a)|2a|_p^{-1/2}\chi_p\big(-\xi^2/4a\big), \quad a \neq 0. \tag{17.41}$$

$$\chi_p(x^2/2) \iff \chi_p(-\xi^2/2), \quad p \neq 2. \tag{17.42}$$

$$\chi_2(x^2/2) \iff \exp(i\pi/4)\chi_2(-\xi^2/2), \quad p = 2. \tag{17.43}$$

$$\sqrt{|a|_p}\exp(-|x|_p^2)\chi_p(ax^2) \iff S(|a|_p^{-1}, 1/p)\chi_p\big(-\xi^2/4a\big)\Omega(|a|_p^{-1/2}|\xi|_p)$$
$$+\Big\{\lambda_p(a)\exp(-|\xi/a|_p^2)\chi_p\big(-\xi^2/4a\big) + |a|_p^{1/2}|\xi|_p^{-1}\big[S(|\xi|_p^{-2}, 1/p)$$
$$-\exp(-|p\xi|_p^{-2})\big]\Big\}[1 - \Omega(|a|_p^{-1/2}|\xi|_p)], \quad p \neq 2, \gamma(a) = 2k. \tag{17.44}$$

$$\sqrt{|a|_p}\exp(-|x|_p^2)\chi_p(ax^2) \iff \Big\{1/\sqrt{p}S(p^{-1}|a|_p^{-1}, 1/p) + [\lambda_p(a) - 1/\sqrt{p}]$$
$$\times\exp(-|pa|_p^{-1})\Big\}\chi_p\big(-\xi^2/4a\big)\Omega(\sqrt{p}|a|_p^{-1/2}|\xi|_p)$$
$$+\Big\{\lambda_p(a)\exp(-|\xi/a|_p^2)\chi_p\big(-\xi^2/4a\big) + |a|_p^{1/2}|\xi|_p^{-1}\big[S(|\xi|_p^{-2}, 1/p)$$

$$-\exp(-|p\xi|_p^{-2})]\Big\}[1-\Omega(\sqrt{p}|a|_p^{-1/2}|\xi|_p)], \quad p\neq 2, \gamma(a)=2k+1. \quad (17.45)$$

$$\sqrt{|a|_2}\exp(-|x|_2^2)\chi_2(ax^2) \iff \Big\{[\sqrt{2}\lambda_2(a)-1]\exp(-|4a|_2^{-1})$$

$$+S(|a|_2^{-1},1/2)\Big\}\chi_2\big(-\xi^2/4a\big)\Omega(|4a|_2^{-1/2}|\xi|_2)+\Big\{\exp(-|4a|_p^{-1})$$

$$+[\sqrt{2}\lambda_2(a)-1]S(|a|_2^{-1},1/2)\Big\}\delta(|\xi|_2-|a|_2^{1/2})\chi_2\big(-\xi^2/4a\big)$$

$$+\Big\{\sqrt{2}\lambda_2(a)\exp(-|2a|_2^{-2}|\xi|_2^2)\chi_2\big(-\xi^2/4a\big)+|a|_2^{1/2}|\xi|_2^{-1}[S(|\xi|_2^{-2},1/2)$$

$$-2\exp(-|2\xi|_2^{-2})]\Big\}[1-\Omega(|a|_2^{-1/2}|\xi|_2)], \quad p=2, \gamma(a)=2k. \quad (17.46)$$

$$\sqrt{|a|_2}\exp(-|x|_2^2)\chi_2(ax^2) \iff 1/\sqrt{2}[S(|a/2|_2^{-1},1/2)-\exp(-|2a|_2^{-1})$$

$$+2\lambda_2(a)\exp(-|8a|_2^{-1})]\chi_2\big(-\xi^2/4a\big)\Omega(|8a|_2^{-1/2}|\xi|_2)$$

$$+\sqrt{2}\Big[S(|2a|_2^{-1},1/2)+\lambda_2(a)\exp(-|8a|_2^{-1})\Big]\chi_2\big(-\xi^2/4a\big)\delta(|\xi|_2-|2a|_2^{1/2})$$

$$+\sqrt{2}\lambda_2(a)S(|2a|_2^{-1},1/2)\chi_2\big(-\xi^2/4a\big)\delta(|\xi|_2-\sqrt{2|a|_2}$$

$$+\Big\{|a|_2^{1/2}|\xi|_2^{-1}[S(|\xi|_2^{-2},1/2)-2\exp(-|2\xi|_2^{-2})]$$

$$+\sqrt{2}\lambda_2(a)\exp(-|2a|_2^{-2}|\xi|_2^2)\chi_2\big(-\frac{\xi^2}{4a}\big)\Big\}$$

$$\times[1-\Omega(2^{-1/2}|a|_2^{-1/2}|\xi|_2)], \quad p=2, \gamma(a)=2k+1. \quad (17.47)$$

$$|x|_p^{\alpha-1}\theta(x) \iff \Gamma_p(\pi_{\alpha,\theta})|\xi|_p^{-\alpha}\theta^{-1}(\xi), \quad \theta\not\equiv 1, \alpha\in\mathbb{C}. \quad (17.48)$$

$$\theta(p^kx)\delta(|x|_p-p^k) \iff p^ka_{p,k}(\theta)\theta^{-1}(\xi)\delta(|\xi|_p-1), \quad k=\rho(\theta) \quad (17.49)$$

where quantity $a_{p,k}$ is defined in (8.17).

$$|z\bar{z}|_p^{\alpha-1} \iff \Gamma_{p,d}(\alpha)|\zeta\bar{\zeta}|_p^{-\alpha},$$

$$\alpha\neq\alpha_k, k\in Z, d\notin\mathbb{Q}_p^{\times 2}, \quad (\text{ see } (9.7)). \quad (17.50)$$

$$|x|_p^{\alpha-1}\mathrm{sgn}_{p,d}x \iff \tilde{\Gamma}_p(\alpha)|\xi|_p^{-\alpha}\mathrm{sgn}_{p,d}\xi,$$

$$\alpha\neq\alpha_k-\pi i/\ln p, k\in Z, p\neq 2, |d|_p=1, d\notin\mathbb{Q}_p^{\times 2} \quad (\text{ see } (8.8)). \quad (17.51)$$

$$|x|_p^{\alpha-1}\mathrm{sgn}_{p,d}x \iff \pm p^{\alpha-1/2}\sqrt{\mathrm{sgn}_{p,d}(-1)}|\xi|_p^{-\alpha}\mathrm{sgn}_{p,d}\xi,$$

$$p \neq 2, |d|_p = 1/p \quad \text{(see (8.24))}. \tag{17.52}$$

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